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PROCEEDINGS

OF THE

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From December 1916 *to August* 1917.

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PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1916-1917.

October 27, 1916.

Meeting held at Imperial College of Science.

Mr. F. E. Smith, A.R.C.S., Vice-President, in the Chair.

The following Papers were read :—

1. " The Application of the Kerr Effect to the Determination of the Saturation Values for Magnetism of Ferromagnetic Metals, Compounds and Alloys." By S. G. Barker, D.I.C., Ph.D.
2. " The Influence of the Time Element on the Resistance of a Solid Rectifying Contact." By D. Owen, B.A., B.Sc.

November 10, 1916.

Meeting held at the Imperial College of Science.

Prof. C. V. Boys, F.R.S., President, in the Chair.

The following Papers were read :—

1. " On Diffusion in Liquids." By B. W. Clack, M.Sc.
2. " On the Regularity in the Distribution of the Satellites of Spectrum Lines, with a Note on the Structure of the Green Line of Mercury ; and Terms of Correction in using a Concave Grating." By Prof. H. Nagaoka.

November 24, 1916.

Meeting held at the Imperial College of Science.

Prof. C. V. Boys, F.R.S., President, in the Chair.

The following Papers were read :—

1. "On the Measurement of the Thomson Effect in Wires." By H. Redmayne Nettleton, M.Sc.
2. "On the Thermo-electric Properties of Fused Metals." By C. R. Darling, A.R.C.S., F.I.C., and A. W. Grace.

January 26, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. Boys, F.R.S., President, in the Chair.

The following Papers were read :—

1. "A Clock of Precision." By C. O. Bartrum, B.Sc.
2. *"The Effect of Water-vapour in the Atmosphere on the Propagation of Electromagnetic Waves." By Dr. Frederick Schwers.

* Taken as read on account of lateness of hour.

Annual General Meeting.

February 9, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. Boys, F.R.S., President, in the Chair.

The Report of the Council was read by the Secretary.

In the year 1916 twelve ordinary meetings have been held. The average attendance at the meetings was 24.

The second Guthrie Lecture, which was to have been delivered in October, 1915, was delivered at the meeting of Jan. 28, 1916, by Mr. W. B. Hardy, M.A. The subject of the lecture was, "Some Properties of Living Matter."

The Annual Exhibition of Apparatus was again suspended owing to the war.

The Board of Invention and Research of the Admiralty have made considerable use of the organisation of the Society. About a dozen Fellows of the Society, apart from those who are permanent members of the panels of the Board, have undertaken tasks and submitted reports. The Board has sent letters of thanks for the work done.

The number of Honorary Fellows on the roll on Dec. 31, 1916, was 11, and the number of Ordinary Fellows was 425. Eight new Fellows and one Student have been elected and there have been three resignations.

The Society has to mourn the loss of one Honorary Fellow, namely Prince Boris Galitzin, and of nine Ordinary Fellows, namely, Mr. C. F. Casella, Dr. W. H. Coffin, Lieut. G. B. Dyke (killed in action), 2nd Lieut. F. G. B. Gardner (killed in action), Prof. J. W. Judd, Sir W. Ramsay, Prof. S. P. Thompson (past President), and Prof. A. M. Worthington. The decease of Dr. H. Müller, who died in 1915, but was not mentioned in the last report, must also be recorded.

The Treasurer presented his Report and Balance-sheet.

The total income of the Society is practically the same as last year. There is a slight decrease in subscriptions ; on the other hand, the sum of £38. 17s. 3d. is recovered income tax.

The expenditure for the year is £88. 4s. 8d. less than in 1915. The income of the Society exceeds the expenditure by a little over £300, which shows that the Society is still sound financially.

I purchased £400 5 per cent. Exchequer Bonds in July, 1916.

There has been a further depreciation in the value of Securities.

Arrears in subscriptions are very serious this year.

The liabilities on account of the Life Compositions Fund are lower by £110, owing to the decease of eleven Life Fellows.

The balance of the General Fund has decreased £200, mainly owing to the reduction in the value of Securities.

Both Reports were put to the meeting and unanimously adopted.

THE TREASURER IN ACCOUNT WITH THE PHYSICAL SOCIETY, FROM JANUARY, 1916, TO DECEMBER 31ST, 1916.

Dr.	£	s.	d.	£	s.	d.
Entrance Fees	8	8	0			
Subscriptions	478	18	1			
Subscriptions paid in Advance	5	0	0			
,, Voluntary	18	18	0			
,, Students'	3	13	6			
,, **Arrears paid**	9	19	0			
,, for "Science Abstracts" and Advance Proofs	23	0	7			
				547	17	2
Composition Fees				Nil.		
Income Tax recovered				38	17	3
Dividends :—						
Furness Debenture Stock	13	1	10			
Midland Railway	32	9	2			
Metropolitan Board of Works	5	10	4			
Lancaster Corporation Stock	9	9	0			
New South Wales	7	0	2			
London, Brighton & South Coast Railway	21	1	2			
Great Eastern Railway	16	7	6			
India 3½% Stock	13	15	10			
Exchequer Bonds, 5%	4	13	0			
				123	8	0
Interest on deposit account				9	14	11
Sales of Publications ("The Electrician" Printing & Publishing Co.)				146	7	9
Transferred from Deposit Account				400	0	0
Balance at Bank, January 1, 1916				115	13	5
				£1,381	18	6

Cr.	£	s.	d.	£	s.	d.
"Science Abstracts"	279	13	6			
Extra copies	3	2	0			
				282	15	6
"The Electrician" Printing & Publishing Co.—						
"Proceedings"	209	5	5			
Bulletin	24	5	9			
Distribution	35	10	4			
General	24	17	10			
	293	19	4			
Less credit for use of type	1	12	0			
				292	7	4
Reprint of Jean's Report				21	5	2
Periodicals	1	15	0			
Reporting	35	0	0			
Refreshments and Attendance	6	0	0			
Petty cash	9	16	11			
Bank Commission and Stamped Cheques		5	8			
Royal Asiatic Society	2	2	0			
International Catalogue	10	0	0			
Guthrie Lecture	10	0	0			
				74	19	7
Phœnix Assurance Co.				9	17	6
				681	5	1
To Deposit Account, April				200	0	0
Invested in Exchequer Bonds, 5%				400	0	0
Balance at Bank, January 1, 1917				100	13	5
				£1,318	18	6

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

J. E. KINGSBURY.
A. A. C. SWINTON.

January 24th, 1917.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31ST, 1916.

ASSETS.	£	s.	d.
Subscriptions due	89	5	0
£533 Furness Railway 3 per cent. Debenture Stock	287	16	0
£1,600 Midland Railway 2½ per cent. Preference Stock	736	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	165	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	238	0	0
£254. 2s. 9d. New South Wales 3½ per cent. Debenture Stock	216	0	0
£500 London, Brighton & South Coast Railway Ordinary Stock	390	0	0
£500 Great Eastern Railway 4 per cent. Debenture Inscribed Stock	370	0	0
£500 India 3½ per cent. Stock	330	0	0
£400 Exchequer Bonds	400	0	0
Balance at Bank	100	13	5
Balance on deposit	100	0	0
Publications (Treasurer's Estimate)	220	0	0
	£3,642	14	5

LIABILITIES.	£	s.	d.
Life Compositions	1,900	10	0
Balance General Fund	1,742	4	5
	£3,642	14	5

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,

J. E. KINGSBURY.
A. A. C. SWINTON.

January, 24th 1917.

LIFE COMPOSITION FUND.

	£	s.	d.
153 Fellows paid £10	1,530	0	0
3 Fellows paid £15	45	0	0
5 Fellows paid £21	105	0	0
7 Fellows paid £31. 10s.	220	10	0
	£1,900	10	0

NOTE.—Five Fellows who paid £10 deceased during year 1916.
Six Fellows who paid £10 deceased during year 1915
(not deducted in last account).

Audited and found correct,

WILLIAM DUDDELL, *Honorary Treasurer.*
January 24th, 1917.

J. E. KINGSBURY.
A. A. C. SWINTON.

After the customary votes of thanks, the election of Officers and Council took place the new Council being constituted as follows :—

President.—Prof. C. Vernon Boys, F.R.S.

Vice-Presidents, who have filled the Office of President.—Prof. G. C. Foster, D.Sc., LL.D., F.R.S. ; Prof. R. B. Clifton, M.A., F.R.S. ; Prof. A. W. Reinold, C.B., M.A., F.R.S. ; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir Oliver J. Lodge, D.Sc., LL.D., F.R.S. ; R. T. Glazebrook, C.B., D.Sc., F.R.S. ; Prof. J. Perry, D.Sc., F.R.S. ; C. Chree, Sc.D., LL.D., F.R.S. ; Prof. H. L. Callendar, M.A., LL.D., F.R.S. ; Prof. A. Schuster, Ph.D., Sc.D., F.R.S. ; Sir J. J. Thomson, O.M., D.Sc., F.R.S.

Vice-Presidents.—W. R. Cooper, M.A., B.Sc. ; Sir Napier Shaw, F.R.S. ; S. W. J. Smith, M.A., D.Sc., F.R.S. ; W. E. Sumpner, D.Sc.

Secretaries.—Prof. W. Eccles, D.Sc. ; R. S. Willows, M.A., D.Sc.

Foreign Secretary.—R. T. Glazebrook, C.B., D.Sc., F.R.S.

Treasurer.—W. Duddell, F.R.S.

Librarian.—S. W. J. Smith, M.A., D.Sc., F.R.S.

Other Members of Council.—H. S. Allen, M.A., D.Sc. ; Prof. E. H. Barton, D.Sc., F.R.S. ; Prof. G. W. O. Howe, D.Sc. ; Prof. J. W. Nicholson, M.A., D.Sc. ; C. C. Paterson ; C. E. S. Phillips, F.R.S.E. ; Prof. O. W. Richardson, M.A., D.Sc., F.R.S. ; S. Russ, M.A., D.Sc. ; T. Smith, B.A. ; F. J. W. Whipple, M.A.

After the conclusion of the General Business the following Papers were read :—

1. " Note on the Calculation of the Coefficient of Diffusion of a Salt at a Definite Concentration." By A. Griffiths, D.Sc.

2. " A Special Test on the Gravitation Temperature Effect." By Dr. P. E. Shaw and Mr. C. Hayes.

March 9, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. BOYS, F.R.S., President, in the Chair.

The following Papers were read :—

1. "To Measure the Pressure in a High Vacuum by Observation of Logarithmic Decrement." By P. E. SHAW, B.A., D.Sc.
2. "A Diffraction Colour Box." By A. W. CLAYDEN, M.A.

Mr. CLAYDEN also showed an arrangement for projecting on a screen the fringes obtained when a Thorpe grating is laid with its ruled face against a plane mirror.

Prof. W. M. COLEMAN exhibited an apparatus for studying the effect of Hertzian waves on the heart.

March 23, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. BOYS, F.R.S., President, in the Chair.

The THIRD GUTHRIE LECTURE was delivered by Prof. P. LANGEVIN, who took as his subject "Molecular Orientation."

April 27, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. BOYS, F.R.S., President, in the Chair.

The following Papers were read :—

1. "A Note on the Derivation of the General Equation of Wave Motion in an Elastic Medium." By Prof. J. A. FLEMING, M.A., D.Sc., F.R.S.
2. "The Effect of Stretching on the Thermal Conductivity of Wires." By A. JOHNSTONE, B.Sc.
3. *"Cohesion" (Third Paper). By Prof. H. CHATLEY, D.Sc.

* Taken as read in absence of author.

May 25, 1917.

Meeting held at the Imperial College of Science.

Mr. W. R. **Cooper**, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "An Investigation of Radium Luminous Compound." By C. C. **Paterson**, J. W. T. **Walsh** and W. F. **Higgins**.
2. "The Resistance to the Motion of a Lamina Cylinder or Sphere in a Rarefied Gas." By F. J. W. **Whipple**.
3. *"The Effect of Stretching on the Thermal and Electrical Conductivities of Wires." By Prof. C. H. **Lees**, F.R.S.

* Taken as read on account of lateness of the hour.

June 8, 1917.

Meeting held at the Imperial College of Science.

Prof. C. V. **Boys**, F.R.S., President, in the Chair.

The following Papers were read :—

1. "An alternating-current Bridge Method of Comparing Two Fixed Inductances at Commercial Frequencies." By T. **Parnell**, M.A.
2. "On the Wave-lengths and Radiation of Loaded Antennæ." By **Balth. van der Pol**, Junr.

Dr. A. **Griffiths** gave a demonstration of a method of preventing sparking at a rapid make and break, which incidentally produces colloidal platinum.

June 22, 1917.

Meeting held at the Imperial College of Science.

Mr. W. R. **Cooper**, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Determination of Coma from a Central Ray." By T. **Smith**, B.A.
2. "Chromatic Parallax and its Influence on Optical Measurements." By J. **Guild**, A.R.C.S., D.I.C., F.R.A.S.

3. "A Note on the Graphs of Bessel Functions." By B. HAGUE, B.Sc., D.I.C.

Mr. S. D. CHALMERS gave a demonstration of the measurement of differences of curvature by means of monochromatic interference rings, and the use of steel spheres as reference standards.

The following Papers, received too late for reading, were included in the August number of the "Proceedings":—

"The Capacity of an Inverted Cone and the Distribution of its Charge." By Prof. G. W. O. HOWE, D.Sc.

"On the Measurement of Small Inductances and on Power Losses in Condensers." By ALBERT CAMPBELL, B.A.

"The Mechanism of Colour Vision." By J. GUILD, A.R.C.S., D.I.C., F.R.A.S.

"The Use of Monochromatic Interference Rings for the Measurement of Curvature." By S. D. CHALMERS, M.A.

I.—*The Application of the Kerr Effect to the Determination of the Saturation Values for Magnetism of Ferro-magnetic Metals, Compounds and Alloys. By* S. G. Barker, *D.I.C., Ph.D., King Edward VII. Research Scholar.*

Received September 4, 1916.

In the year 1876 Kerr[1] observed that the direction of vibration of plane polarised light was altered when the polarised ray was reflected from a ferro-magnetic mirror placed in a magnetic field. Further work on this phenomenon was done later by Gordon,[2] Fitzgerald,[3] Hall,[4] Kaz,[5] Righi,[6] Kundt,[7] Du Bois,[8] Sissingh,[9] Zeeman,[10] Ingersoll,[11] Foote,[12] Loria,[13] Martin,[14] and many others. It has been conclusively shown that the Kerr effect is only possible in the case of ferro-magnetic bodies such as iron, nickel, cobalt, iron oxides, &c., whilst diamagnetic bodies like bismuth have shown no appreciable effect. As well as the change of the plane of polarisation, yet another change is evident—namely, if the incident light is plane polarised, the reflected ray is elliptically polarised. The major axis of the ellipse corresponds to the new plane of vibration of the plane polarised light. In the simplest case of nearly vertical reflection either perpendicular or parallel to the plane of incidence of the polarised light, and with normal magnetisation, the rotation is dependent only on the wavelength and the strength of the magnetic field. This simple case is used exclusively throughout this work.

The dispersion curves throughout the spectrum have been investigated for the Kerr effect by a large number of observers, amongst whom Du Bois was one of the earliest, followed later

---

(1) J. Kerr, "Phil. Mag." (5), 3, p. 321, 1877; 5, p. 161, 1878.

(2) J. E. H. Gordon, "Phil. Mag." (5), 4, p. 104, 1877.

(3) G. F. Fitzgerald, "Proc." Roy. Soc., 25, p. 447, 1876.

(4) E. H. Hall, "Phil. Mag." (5), 12, p. 171, 1881.

(5) P. C. Kaz, "Diss. Amsterdam," 1871.

(6) A. Righi, "Ann. d. Chem. u. Phys." (6) 4, p. 443, 1885.

(7) A. Kundt, "Wied. Ann.," 23, p. 228, 1884; 27, p. 199, 1886.

(8) H. du Bois, "Wied. Ann.," 39, p. 228, 1890.

(9) R. Sissingh, "Arch. Neerland." (1), 27, p. 173, 1894.

(10) P. Zeeman, "Arch. Neerland.," 27, p. 252, 1894.

(11) L. P. Ingersoll, "Phil. Mag." (6), 11, p. 41, 1906; 18, p. 74, 1909.

(12) P. D. Foote, "Phys. Review" (2), 34, p. 96, 1912.

(13) St. Loria, "Ann. d. Phys.," 38, p. 889, 1912.

(14) P. Martin, "Ann. d. Phys.," 39, p. 625, 1913.

by Ingersoll, Loria, Martin and others. Ingersoll has confined his attention to the infra-red part of the spectrum and used wave-lengths between 1μ and 3μ. He accomplished this by means of a bolometer method. Du Bois investigated the curves for iron, nickel, cobalt and magnetite over the whole visible spectrum. In the present work the curves of Du Bois have been more accurately re-determined, and the number of points taken considerably amplified. These will be discussed later. Du Bois found that iron gave a negative rotation from the violet to the red with a probable minimum in the ultra-violet. Nickel gave a minimum in the yellow, and cobalt a minimum in the blue-green. Magnetite exhibited a positive maximum rotation in the yellow, and a zero value in the violet. Ingersoll prolonged these curves into the infra-red. He showed that for iron and cobalt there are well-defined negative maxima near $\lambda=1\mu$, and then quick decreases in numerical value. Nickel showed a more remarkable phenomenon in that between the wave-lengths $\lambda=1\mu$ and $\lambda=1{\cdot}5\mu$ the value passed through zero. Magnetite exhibited two inversion points. Du Bois has closely investigated the relation between the magnetic field strength and the Kerr rotation, and established the Kerr constant for various metals. Foote[*] examined the curves for ellipticity and dispersion. Loria and von Zakrzewski[†] attempted to find a relation between the Kerr effect and the optical constants of the metals. Chemists and metallurgists have placed us to-day in possession of many prepared ferro-magnetic substances. Hilpert[‡] has produced metaferrites by combinations of different oxides with iron oxide. In all cases in this series he has maintained the identical structure of magnetite, FeO, Fe_2O_3, using the latter part as the acid radicle and simply substituting CuO, &c., for FeO. The iron parts of the compounds evidently carry the magnetic properties.

The object of the present work is to demonstrate the theory of du Bois[§] that in the case of ferro-magnetic substances there exists a proportionality between the rotation (ε) and the intensity of magnetisation (I). When the maximum magnetisation is reached (I_m), the rotation (ε_m) will become constant. According to Du Bois, if the rotation be plotted as a function of the magnetising field—*i.e.*, $\varepsilon = F(H)$, then the value of I_m can be found. In the case of an infinite homogeneous plate, with normal magnetisation, the point of intersection of the straight line

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[*] P. D. Foote, "Phys. Review" (2), 34, p. 96, 1912.

[†] St. Loria and C. Zakrzewski, "Bull." de l'Académie des Sciences de Cracovie (A), 22, p. 275, 1910.

[‡] S. Hilpert, "Ber. Deutsch. Chem. Ges.," 42, p. 2248, 1909.

[§] H. du Bois, "Wied. Ann.," 31, p. 952, 1887; "Phil Mag.," 29, p. 301, 1890.

$$\varepsilon = K.I. = K.H./4\pi$$

and the asymptote $\varepsilon = \text{constant}$

gives an abscissa (x) whose value is $4\pi I_m$, where I_m is the saturation value. Loria began preliminary investigations of this point, but his results obtained with samples of quite irregular configuration were not at all concordant. In the present work specimens of iron, nickel and cobalt, and many other substances whose saturation values had been carefully determined, were taken as the groundwork on which to base the efficiency of the method. These specimens were all of the standard shape discussed later, and were investigated first of all. The values so obtained were in excellent agreement with those found by other methods. Later in the work various other materials of unknown saturation values were examined. The great advantage of this method is that the saturation value of a substance can be found when only a very small quantity is available. The mirrors used in all the tests were of the uniform size 2·5 mm. in radius and 0·5 mm. thick. The method of procedure and the description of the apparatus is set out below. The curves plotted show ε as $F(H)$, and the sign of the rotation is noted in every case. Loria found in his work that wave-length had an influence on the saturation value obtained. With the same material, using light of different wave-lengths, he obtained different saturation values from the different curves obtained. This work shows, as would be expected, that he was wrong, and that wave-length has no influence on the values obtained. His error was probably due to the irregular shape of his mirrors and the very few and insufficient number of points taken.

Description of Apparatus Used.

The apparatus used is illustrated in the accompanying diagram. The light from an arc lamp (in exceptional cases from the sun) was focussed on the slit of a Du Bois* monochromator

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* H. du Bois, "Zeitshr. f. Instrumenten.," 31, p. 1, 1911.

chromator, the slit of which was altered in width according to the intensity of illumination desired for any particular wave-length. In the extreme blue and red regions the slit was, with safety, made much wider, but in the green $\lambda = 530 \cdot 0 \mu\mu$ the width was uniformly kept at 0·15 mm. The light passed through the monochromator and emerged from the same through a slit of equal width to the one on which the light was incident. It was then immediately focussed on to a Du Bois half-shade polarimeter.* In this instrument the beam is reflected through an angle of 90 deg. by a mirror. It next passes through two Nicol's prisms and emerges as plane polarised light. The polarised beam passes onwards through a hole in the arm of a large Du Bois ring magnet, and emerges through the pole-piece by a vertical rectangular slit. From this it is incident on a small metallic mirror placed on the other pole-piece a small distance away. The return beam is received in the analyser, and the rotation is measured on the vertical scale by an optical arrangement as shown in the diagram. This system of taking readings enabled extremely small differences in the rotation to be measured with considerable accuracy. The light from an illuminated scale was reflected by a plane mirror on to another plane mirror attached to the analyser, and which rotated with it. From this it was reflected through a right-angled prism into a large magnifying telescope, by means of which the readings could easily be taken with great accuracy. The angle between the Nicol's prisms was 0·4 of a degree, and the mean error of reading about 0·01 of a minute. In certain exceptional cases the light was slightly elliptically polarised, but in no case to an extent sufficient to cause annoyance or error. The field was obtained by a large Du Bois ring magnet† of resistance 1·5 ohms. After applying strong currents it was found necessary to open the magnetic circuit and demagnetise the instrument by knocking. The intensity of the magnetic field was measured primarily by a ballistic method, and subsequently by means of a thin glass etalon with silvered back surface. The etalon was placed between the poles and the rotation of the plane of polarisation, when different fields were existing was measured as before. The etalon was always placed close to the metallic mirror—*i.e.*, the back pole. Owing to the reflection from the silvered back

* H. du Bois "Wied. Ann.," 46, p. 545, 1892.

† H. du Bois, "Wied. Ann.," 51, p. 549, 1894.

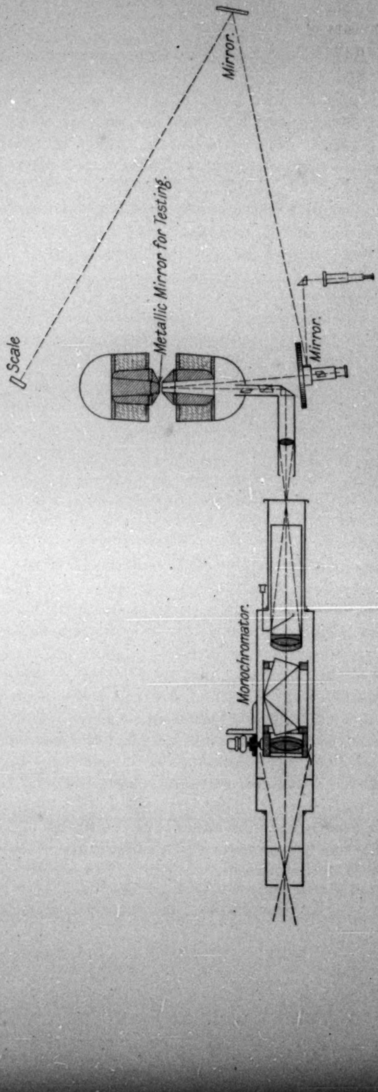
Mirror.
Metallic Mirror for Testing.
Scale
Mirror.
Monochromator.

DIAGRAM OF APPARATUS.

surface, the light passed twice through the glass, and therefore the rotation observed was double the proper value. On commutating the current in the magnet coils the resultant rotation was four times the simple rotation, and for wave-length $\lambda = 530{\cdot}0 \mu\mu$ was $4 \times 3{\cdot}26$ minutes per kilogauss. The etalon had a mean thickness of 0·35 mm. and an angle of slightly less than 20 minutes. According to the theory of polar armatures,* using a vertical slit and with a polar distance of 1·6 mm. the field in the back part of the air space should be practically uniform. This uniformity was tested for all mirrors used, and the results for cobalt and nickel are appended below. A steady current of 20 amperes was maintained in the magnet, and light of wave-length $530{\cdot}0 \mu\mu$ was used. The etalon was placed in various positions across the air space and the rotation determined in each case. In all the experiments performed the field was found to increase rapidly from the slit towards the mid-point of the pole space, and then became constant in the neighbourhood of the metallic mirror. By this means it was experimentally determined that the field immediately in front of the mirror was uniform and capable of measurement to within 1 per cent. by the etalon. The correction applied by Foote in his work was therefore unnecessary. Tables I. and II. give the results of experiment with cobalt and nickel mirrors to show the increase in the value of the intensity of the magnetic field across the air space and the constancy of the same in the back half of that space.

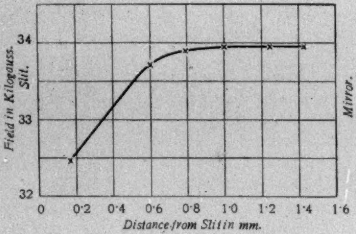

FIG. 1.—CONSTANCY OF FIELD COBALT MIRROR.

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* H. du Bois, "Ann. d. Physik.," 42, p. 911, 1913.

TABLES I. AND II.

Distance of etalon from slit.	Cobalt mirror. Field in kilogauss.	Nickel mirror. Field in kilogauss.
0·17 mm.	32·48	30·65
0·60 mm.	33·71	31·88
0·80 mm.	33·89	32·30
1·05 mm.	33·94	32·48
1·25 mm.	33·94	32·48
1·43 mm.	33·94	32·48

All distances were measured to centre of etalon.

The mirrors for the measurement of the Kerr effect were mounted on cones of iron and inserted in a conical space in the pole-piece. The mirror itself consisted of a plane disc of circular dimensions, 2·5 mm. in radius and 0·5 mm. thick. This was the same in all cases. The mirrors were soldered on to the cones and then polished to give a plane and highly reflecting surface. In order to obtain uniformity of field (by Du Bois theory), the mirror was always placed at a distance of 1·6 mm. from the slit. It was thought that the air in the space between the pole-pieces might be a possible source of error in enhancing the rotation, so a silver mirror was inserted between the poles. As the result of 100 independent readings of the rotation by commutating the current in the magnet coils, it was found that for the highest fields, using light $\lambda = 506\ \mu\mu$, the rotation was 0·015′, and consequently within the limits of experimental error. After each set of readings for the saturation curve the etalon calibration was immediately taken and the field strengths calculated. There was no visible increase in the rotation due to the etalon for currents above 20 amperes.

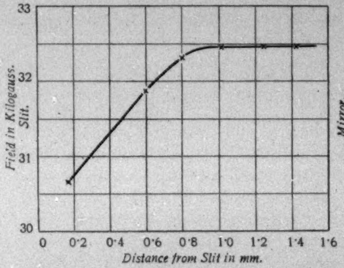

FIG. 2.—CONSTANCY OF FIELD NICKEL MIRROR.

Throughout the whole Paper the following notation will be used for the various quantities :—

H=Field intensity in kilogauss.
I_m=Saturation value in C.G.S.
ε_m=Maximum rotation.
K=Kerr's constant.
I=Magnetisation.
ε=Rotation in minutes of arc.
λ=Wave-length used in 10^{-6} mm.
s_m=Specific magnetisation.

The sign of the rotation is taken according to the usual convention that in the case of iron it is negative. Below are appended :—

1. The results of the determination of saturation values of materials which have been tested by other methods. This was done to justify the method.
2. Determinations of saturation values of ferro-magnetic metals and alloys.
3. Dispersion curves of these substances which have been more accurately determined than before.

EXPERIMENTS WITH METALS OF KNOWN VALUES.

Electrolytic Iron.—(1) This material was kindly given to me by Dr. Gumlich, of the Physische Technische Reichsanstalt, Berlin, in 1914, and was a piece of iron the saturation value of which he had determined by the isthmus method. He gave the value as 1,705 C.G.S. units. The results are set out in Table III.

TABLE III.

Saturation curve. Iron (1). λ=530·0 $\mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes
2·34	2·54	21·74	21·77
4·00	4·20	23·19	22·20
5·68	6·00	24·81	22·26
7·60	8·20	26·51	22·30
9·13	9·80	32·30	22·68
10·80	11·80	35·83	22·68
12·28	13·30	36·56	22·68
14·52	15·90	39·44	2·68
17·51	19·00	40·42	22·68
19·86	20·70	...	...

Rotation was negative throughout.

The results were plotted in Fig. 3, and it is easily seen that the value of x, the abscissa of the point of intersection of the asymptote and the saturation curve horizontal produced

$$x = 4\pi I_m = 21{\cdot}00 \text{ kilogauss.}$$

Therefore $I_m = 1{,}681$ C.G.S. units.

This agrees very well with the value given by Gumlich, the difference being 24, or 1·5 per cent.

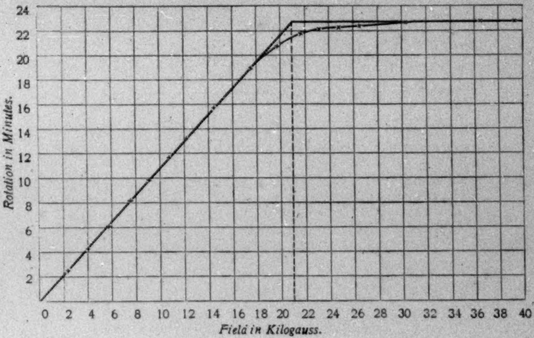

FIG. 3.—IRON (1). SATURATION CURVE.

Dispersion Curve.—The dispersion curve for the Kerr effect was also found for this specimen of iron. There is nothing unusual or extraordinary in it. The curve shows a rapid rise from the blue to the red end of the spectrum. The field used was 36·56 kilogauss, and it is evident from the curve, Fig. 3, that this is above the saturation value. The observations are in agreement with those of Martin and Foote.

Table IV. shows the figures obtained for the dispersion.

TABLE IV.

Wave length in $\mu\mu$.	Rotation in minutes.	Wave length in $\mu\mu$.	Rotation in minutes.
442·5	16·70	546·1	23·73
457·4	17·40	557·5	24·57
474·0	18·27	571·5	25·34
492·5	19·32	599·0	26·33
515·5	20·79	628·0	27·20
531·5	22·54	650·0	27·65
538·5	23·31	690·0	28·56

Rotation was always negative and the field saturated.

Iron.—(2) For this piece of material I have to thank Prof. Weiss,* of Zurich, who had determined its saturation value as $s_m=216{\cdot}1$. The density of the material was 7·86, giving us

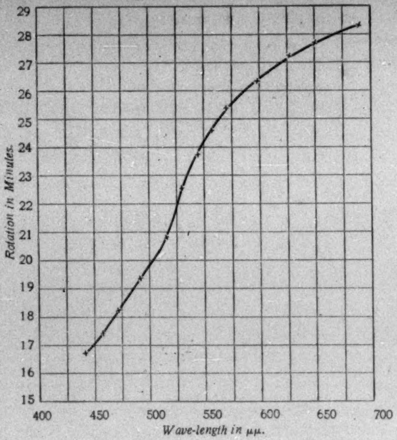

FIG. 4.—IRON (1). DISPERSION CURVE. SPECIMEN SATURATED.

as value for $I_m=1{,}690$ C.G.S. The specimen was free from sulphur, silicon or manganese, and had been melted in an atmosphere of nitrogen. The same process as in the previous case was gone through and the curve drawn. The results gave as the value of the abscissa of the point of intersection

$$x=4\pi I_m=21{\cdot}0 \text{ kilogauss.}$$

Therefore $I_m=1{,}681$ C.G.S.

This value is in excellent agreement with that of Prof. Weiss, the error being 9 C.G.S., or 0·5 per cent.

Iron.—(3) This specimen of Swedish iron was supplied by the Kolswa Iron Works, and contained 0·22 per cent. of carbon, phosphorus, silicon and manganese. Prof. Weiss again supplied me with his calculation of the saturation value, and gave

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* P. Weiss, "Jour. de Phys." (4), p. 273, 1910.

s_m=216·5 C.G.S. The density was returned as 7·8624, and this gives as value of I_m=1,702 C.G.S.

Two sets of readings were taken, using light of different wave-lengths—namely, λ=546·1 and λ=590·1$\mu\mu$. The saturation value obtained in both cases was the same.

TABLE V. (*a*) *and* (*b*).

Saturation curves. Iron (3). (*a*) λ=546·1 $\mu\mu$. (*b*) λ=590·1 $\mu\mu$.

Field in kilogauss.	Rotation in minutes (*a*).	Rotation in minutes (*b*).
2·89	2·88	3·40
6·00	6·09	7·06
9·60	9·77	11·43
12·40	12·25	14·82
15·43	15·17	18·43
17·51	17·05	21·08
19·67	18·32	22·26
21·71	19·05	23·10
23·37	19·78	23·73
24·91	20·25	24·23
27·07	20·46	24·50
32·65	20·48	24·99
36·18	20·59	25·13
38·08	20·59	25·13
39·70	20·59	25·13

Rotation was always negative.

The value of "x" so obtained was 20·95 kilogauss, which gives

$$I_m = \frac{20{\cdot}950}{4\pi} = \underline{1{,}668 \text{ C.G.S.}}$$

All further tests of the material gave values between 1,668 and 1,681 C.G.S. Taking the mean of all values found as 1,675 C.G.S. (being the mean of 25 curves), we find this in good agreement with the other value given above of 1,702 C.G.S., the error being only 1·4 per cent.

Dispersion Curve.—The dispersion curve for this specimen was found, and the results are set out in Table VI. The dispersion throughout the spectrum was fairly large, and from λ=500·0 $\mu\mu$ to λ=700·0 $\mu\mu$ the curve follows practically a straight line. The smaller values of the wave-length show a slight inflection of the curve. It is interesting to compare the values obtained in the present work with those obtained by Foote for a similar specimen. The values are compared in Table VI.(*a*), and are illustrated in Fig. 6. Table VI. gives the results of experiment in the determination of the dispersion curve. In this case the arc lamp was not used as the source of light, but direct sunlight was focussed on to the slit of the monochromator. This additional intensity of the illumination enabled the readings to be taken much further into the violet than before. In all cases the maximum value of the rotation is given.

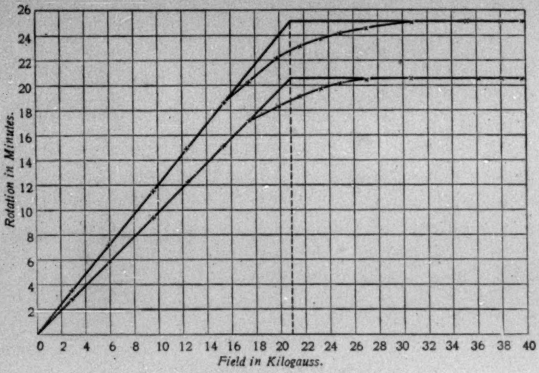


FIG. 5.—IRON (3).

Upper Curve (A) Wave-length 546, 1μμ (A).
Lower Curve (B) Wave-length 590, 1μμ (B).

TABLE VI.

Dispersion curve. Iron (3). Field strength, 39·70 kilogauss.

Wave-length in μμ.	Rotation in minutes.	Wave-length in μμ.	Rotation in minutes.
435·0	14·66	531·5	20·14
442·5	14·95	538·5	20·55
457·4	15·55	546·1	20·93
465·5	15·85	571·0	22·79
478·5	16·36	599·0	24·33
492·5	17·61	650·0	27·47
504·0	18·55	690·0	29·75
515·5	19·15	...	...

The field was saturated.

Table VI. (*a*).—*Comparison of the Values of Barker and Foote.*

Wave-length in $\mu\mu$.	Rotation in minutes (Barker)	Rotation in minutes (Foote).	Ratio.
410·0	14·60	15·00	1·02
440·0	15·00	15·60	1·03
480·0	16·65	16·80	1·01
520·0	19·50	19·74	1·01
560·0	22·10	22·74	1·02
600·0	24·45	24·78	1·01
640·0	26·60	26·58	0·999
660·0	27·94	26·94	0·964

Taking as the mean ratio 1·01, we see that the two sets of values are in good agreement. The present work has found points further into the violet and red than that of Foote, and also more points were obtained. Many other specimens of iron were tested with satisfactory results.

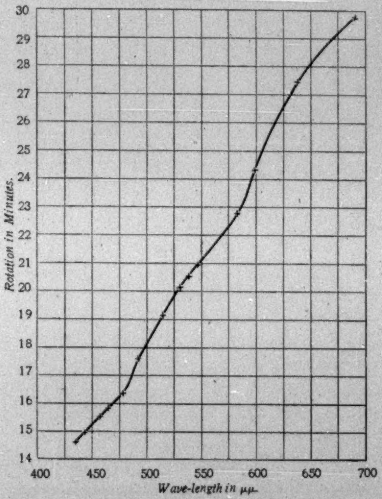


Fig. 6.—Iron (3). Dispersion Curve. Field Saturated.

COBALT.

Cobalt.—(1) This specimen of electrolytic cobalt was provided by Merck, and had been fused in nitrogen. Prof. Weiss has again supplied me with the information regarding his tests of the material for the saturation value. He gave the value as 1,412 C.G.S. The piece was mounted on a cone and tested in the same way as the iron. The results are given in Table VII. and plotted in Fig. 7.

TABLE VII.

Saturation curve. Cobalt (1). $\lambda = 530{\cdot}0\ \mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
2·40	2·85	20·28	20·30
4·76	5·61	21·50	20·45
7·06	8·37	22·43	20·54
9·75	11·44	27·04	20·58
12·13	14·19	30·26	20·58
13·59	15·95	31·64	20·58
15·67	18·20	33·02	20·58
17·82	19·65	33·88	20·58
19·04	20·27	...	...

Rotation was negative throughout.

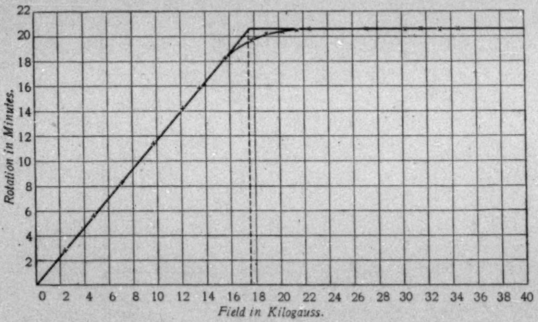


FIG. 7.—COBALT (1). SATURATION CURVE. ROTATION NEGATIVE.

From the curve it is seen that

$$x = 4\pi I_m = 17{,}600 \text{ kilogauss.}$$

Therefore $$I_m = \frac{17{,}600}{4\pi} = 1{,}400 \text{ C.G.S.}$$

This value (1,400) is in excellent agreement with that of Weiss—namely, 1,412 C.G.S., the error being less than 1 per cent. The foregoing affords another example of the accuracy of the method for measuring high saturation values.

Dispersion Curve.—The dispersion curve was observed for the first time for a saturated specimen. It gave no indications of anything other than that previously observed by Martin for cobalt mirrors which were unsaturated. The definite but flat minimum in the green was more closely observed and located exactly. A great number of points were taken, and the values so obtained are set out in Table VIII. The field was maintained at 35·91 kilogauss throughout the whole experiment, and light from the arc lamp was used.

TABLE VIII.

Dispersion curve. Cobalt (1). Field strength, 35·91 kg.

Wave-length in $\mu\mu$.	Rotation in minutes.	Wave-length in $\mu\mu$.	Rotation in minutes.
442·5	23·60	546·1	22·36
457·2	23·37	571·5	22·59
492·5	22·80	599·0	22·96
515·5	22·45	618·0	23·24
525·0	22·31	650·0	23·68
531·5	22·30	690·0	24·25
538·5	22·30	...	...

Rotation was negative throughout.

The values given above bear an interesting comparison with those obtained by Martin. If his results be multiplied by the factor 1·134 the above curve is obtained. The curve is shown in Fig. 8. Table VIII.(*a*) gives the comparison of Martin's values with those above.

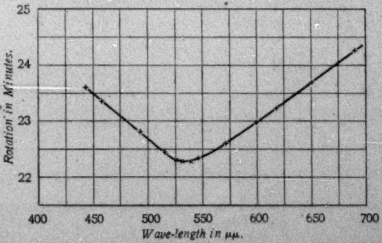


FIG. 8.—COBALT (1). DISPERSION CURVE. ROTATION NEGATIVE.

TABLE VIII. (*a*).

Wave-length in $\mu\mu$.	Rotation in minutes (Barker).	Rotation in minutes (Martin).	Ratio.
483·0	22·95	20·20	1·131
530·0	22·30	19·74	1·130
567·0	22·65	19·92	1·137
615·0	23·20	20·28	1·139
675·0	24·00	21·15	1·135

Allowing for the fact that the Martin observations were taken with an unsaturated mirror and in this case it was saturated, the agreement is good throughout.

Cobalt.—(2) Another specimen was similarly tested and was supplied to me through the kindness of Prof. Merrett from the Metallurgical Department of the Imperial College, London. It was considered tolerably pure. The whole apparatus was maintained as before. The value so obtained from the curve was 1,395 C.G.S., which is again in good agreement with Weiss' result for cobalt. Other specimens were tested and gave equally satisfactory results.

NICKEL.

Nickel.—(1) This piece was originally supplied by Merck and afterwards fused in nitrogen by Weiss. Prof. Weiss gave as its saturation value 490 C.G.S. The mirror was prepared and tested in the same way as before, and the curve followed the usual form. The results of experiment are shown in Table IX.

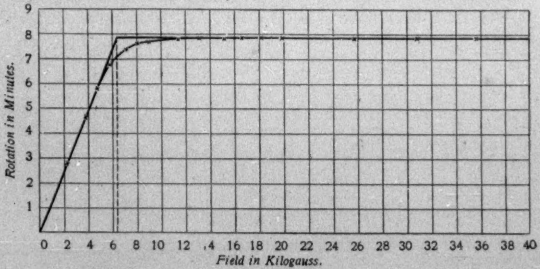


FIG. 9.—NICKEL (1). SATURATION CURVE. ROTATION NEGATIVE.

TABLE IX.

Saturation curve. Nickel (1). $\lambda=590{\cdot}0\ \mu\mu$.

Field in kilo-gauss.	Rotation in minutes.	Field in kilo-gauss.	Rotation in minutes.
2·38	2·81	11·40	7·80
3·80	4·60	13·21	7·85
4·76	5·72	15·05	7·85
5·70	6·75	16·66	7·85
7·10	7·35	18·16	7·85
8·00	7·60	*19·28	7·85
9·20	7·65	...	...

* All readings were constant for higher fields.

From the curve, Fig. 9, we see that

$$x=4\pi I_m=6{,}400 \text{ kilogauss}.$$

Therefore $$I_m=\frac{6{,}400}{4\pi}=\underline{509 \text{ C.G.S.}}$$

This value is a little higher than those before obtained, but still the difference is less than 4 per cent. The process of soldering to the cone might have had a little influence in raising the value. The method is therefore good for substances ot weaker magnetisation.

Disperison Curve.—Great care was taken to determine the exact form of the dispersion curve, and more than twice as many points as found by Foote were taken. The curve agrees remarkably well with his. A well-defined flat minimum was found in the region from $\lambda=530{\cdot}0\ \mu\mu$ to $\lambda=550{\cdot}0\ \mu\mu$, and a maximum was observed at $\lambda=600{\cdot}0\ \mu\mu$, coming again to a minimum at $\lambda=630{\cdot}0\ \mu\mu$. These irregularities were very carefully observed, and the results were determined with great accuracy. Table X. gives the values found.

TABLE X.

Dispersion curve. Nickel (1). Field strength, 31·40 kg.

Wave-length in $\mu\mu$.	Rotation in minutes.	Wave-length in $\mu\mu$.	Rotation in minutes.
442·5	8·316	557·5	7·196
457·2	8·092	571·5	7·40
474·0	7·80	599·0	8·05
492·5	7·52	618·0	7·90
515·5	7·32	628·0	7·75
531·5	7·21	650·0	8·076
539·0	7·21	690·0	8·230
546·1	7·21	...	...

The nickel was saturated and a field of 31·400 kilogauss was maintained throughout.

Nickel.—(2) This piece was supplied as tolerably pure nickel from the Metallurgical Department of the Imperial College, London, by Assistant Prof. Merrett. It was duly tested as before and gave as maximum rotation 8·05 minutes. From the curve obtained x was found to be equal to 6,450 gauss, and this gives as value of I_m or the saturation value equal to 513 C.G.S. This result is in accordance with the one previously obtained for Nickel (1). Other specimens of nickel were examined and gave consistent results of a like order.

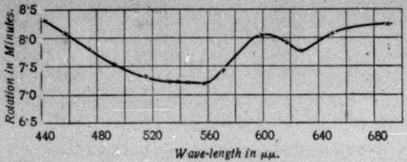


FIG. 10.—NICKEL (1). DISPERSION CURVE. ROTATION NEGATIVE.

The method having thus far been tested for ferro-magnetic metals, it was next applied to alloys of the same. By the kindness of Dr. Gumlich, of the Physische Technische Reichsanstalt, Berlin, specimens of ferro-manganese alloys of known saturation values were obtained. Concordant results were again found and the results are set out below.

ALLOYS.

Ferro-Manganese.—(1) This specimen was kindly supplied by Dr. Gumlich, and had the percentage composition—

Iron = 92·2 per cent.
Manganese = 7·8 per cent.

Dr. Gumlich gave as the saturation value 1,360 C.G.S. With light of wave-length λ=530·0 $\mu\mu$, the maximum rotation was found to be 19·70 minutes, and throughout the whole experiment was in the negative direction. Table XI. gives the values obtained for the saturation curve.

TABLE XI.

Saturation curve. Ferro-Manganese (1). $\lambda = 530 \cdot 0 \, \mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
2·536	3·0	19·320	18·83
4·000	4·7	21·980	19·60
4·602	5·4	23·560	19·70
5·323	6·2	24·090	19·70
7·399	8·8	26·080	19·70
8·303	9·8	31·660	19·70
9·745	11·5	35·330	19·70
11·820	14·0	37·000	19·70
14·890	16·63	39·070	19·70
17·510	18·34	39·610	19·70

The rotation was negative.

From the curve so obtained (Fig. 11) we find that

$$x = 4\pi I_m = 16{,}750 \text{ kilogauss.}$$

Therefore $$I_m = \frac{16{,}750}{4\pi} = \underline{1{,}333 \text{ C.G.S.}}$$

This is in excellent agreement with the value given by Gumlich (1,360), the error being less than 2 per cent.

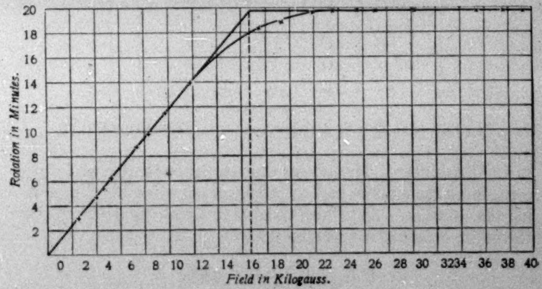


FIG. 11.—FERRO-MANGANESE (1). SATURATION CURVE. ROTATION NEGATIVE.

Ferro-Manganese.—(2) This specimen was also supplied by Dr. Gumlich, and was a piece which he had himself tested by the isthmus method. The usual mirror was prepared and mounted in the large Du Bois magnet as before. The composition of the piece was given by Gumlich as

Iron=89·6 per cent.
Manganese=10·4 per cent.

Light of wave-length $\lambda=530{\cdot}0\ \mu\mu$ was used, the same as in the case of specimen (1). The maximum rotation fell from 19·70 to 13·16 minutes. This shows that the introduction of 2·6 per cent. more manganese had the effect of considerably diminishing the rotation. As before, the rotation was negative. From the curve obtained it was seen that

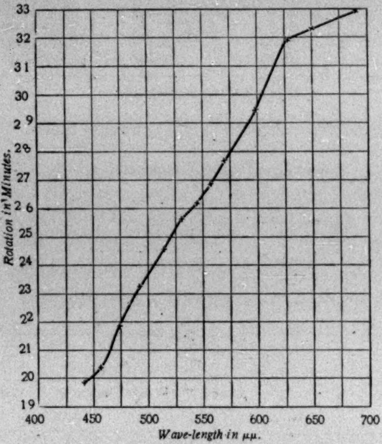


Fig. 12.—Iron-Cobalt Antimony Alloy. Dispersion Curve.

$$x=4\pi I_m=10{,}970 \text{ kilogauss.}$$

Therefore $$I_m=\frac{10{,}970}{4\pi}=\underline{873 \text{ C.G.S.}}$$

Dr. Gumlich gave his value as 885 C.G.S., and by comparison of figures the close agreement of the two methods is readily seen, the error being less than 1·4 per cent.

The foregoing examples are sufficient to demonstrate the efficacy of the method for measuring saturation values of highly magnetic substances. In this case the error from other recognised methods is never more than 2 per cent. When bodies of less magnetic power are tested, then the error is perhaps a little greater, but in no case greater than 4 per cent. The apparatus was now used to determine the saturation values of bodies hitherto untested, and which in many cases could only be obtained in small quantities. In all cases mirrors were prepared as before and highly polished. In some few cases the dispersion curves were also determined. According to the results obtained above the differences between the experimental values and those obtained from the theory of Du Bois are very small. With regard to bodies of low magnetic value, the possible reduction of the percentage error is occupying my attention, and already more accurate readings have been obtained. In the cases below, where the saturation value is below 500 C.G.S., the results can be regarded as correct to within 3 per cent., whilst for bodies of higher values 1 per cent., or at the most 1·5 per cent., is the limit of error.

Ferro-Cobalt-Antimony.—This alloy was supplied from the collection of Prof. Du Bois, and contained

Iron=66 per cent.
Cobalt=30 per cent.
Antimony= 4 per cent.

Using light of wave-length $\lambda=530{\cdot}0\ \mu\mu$, a maximum rotation of 23·66 minutes was obtained in the negative direction. The curve followed the usual form.

The value found for x on plotting the values obtained was

$$x=4\pi I_m=19{,}600 \text{ kilogauss.}$$

Therefore $$I_m=\frac{19{,}600}{4\pi}=\underline{1{,}607 \text{ C.G.S.}}$$

An alloy of these three metals gives, therefore, a high saturation value. The dispersion curve was also determined for this specimen and the result is set out in Table XII.

TABLE XII.

Dispersion curve. Alloy Fe-Co-Sb. Field=32·94 kg.

Wave-length in $\mu\mu$.	Rotation in minutes.	Wave-length in $\mu\mu$.	Rotation in minutes.
442·5	19·74	557·5	26·81
457·4	20·33	571·5	27·68
474·0	21·81	599·0	29·43
492·5	23·24	628·0	31·92
515·5	24·57	650·0	32·30
531·5	25·65	690·0	32·52
546·1	26·18	...	...

Rotation is negative.

The curve is quite regular, and shows a steady increase from $\lambda=440$ $\mu\mu$ to $\lambda=610{\cdot}0$ $\mu\mu$. After this point is reached it bends sharply and shows a very small increase from $\lambda=610$ $\mu\mu$ to $\lambda=700$ $\mu\mu$. There is a tendency towards a minimum value in the violet and towards a maximum in the red.

Ferro-Cobalt.—This was given me by Prof. Weiss, and was stated to be a compound of the formula Fe_2CO. It was the outcome of a fusion of the two metals by Weiss and von Freudenreich. Using $\lambda=530{\cdot}0$ $\mu\mu$, the maximum negative rotation was found to be 27·64 minutes. The curve obtained gave x equal to 23·300 kilogauss, and this furnishes us with a saturation value of 1,850 C.G.S. Both the rotation and the saturation values found were high.

Ferro-Nickel.—This was also kindly supplied by Prof. Weiss from the same source as the ferro-cobalt. It had the definite composition shown by the formula Fe_2Ni. The material gave an excellently polished mirror, and using light of wave-length $\lambda=530{\cdot}0$ the maximum negative rotation was found to be equal to 15·05 minutes. The results are shown in Table XIII.

TABLE XIII.

Saturation curve. Ferro-Nickel. $\lambda=530{\cdot}0$ $\mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
2·436	3·28	10·74	13·86
3·835	5·40	12·81	14·56
5·234	7·20	15·52	14·86
7·038	9·68	17·14	15·05
8·121	11·21	19·13	15·05
9·204	12·50	...	...

Rotation is negative. Rotation is constant for all higher fields.

This curve, when plotted as in Fig. 13, gives $x = 11{\cdot}00$ kilogauss, and this gives as the saturation value $I_m = 875$ C.G.S.

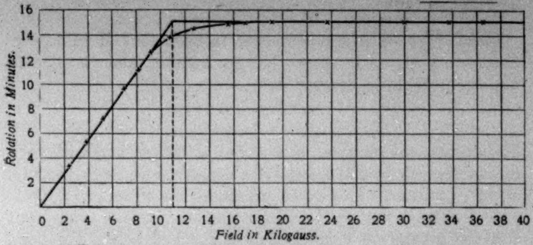


FIG. 13.—FERRO-NICKEL. SATURATION CURVE. ROTATION NEGATIVE.

Cobalt-Silicon.—This alloy was prepared at the Metallurgical Laboratory of the Imperial College, London, by kind permission of Asst. Prof. Merrett. It had the percentage composition—

Cobalt = 98 per cent.
Silicon = 2 per cent.

Using light of wave-length $\lambda = 530{\cdot}0\ \mu\mu$, a maximum negative rotation was attained of 20·52 minutes. A regular curve was given, and the point of intersection showed $x = 13{\cdot}80$ kilogauss. Thus we have as the saturation value

$$I_m = 1{,}094 \text{ C.G.S.}$$

Cobalt-Chromium.—The specimen was prepared as above in London and had the composition—

Cobalt = 87 per cent.
Chromium = 13 per cent.

With all conditions as before, the substance gave a maximum negative rotation of 13·72 minutes. From the graph so obtained x was found to be equal to 11·22 kilogauss, given as the saturation value

$$I_m = 880 \text{ C.G.S.}$$

Cobalt-Chromium-Nickel.—Prof. Tammann, of the University of Gottingen, sent me the piece tested, and kindly gave me the analysis of its composition as

Cobalt = 83 per cent.
Chromium = 15 per cent.
Nickel = 2 per cent.

The curve was of the usual order, and gave a negative maximum rotation of 8·96 minutes. By deduction therefrom the value of $x=9{\cdot}75$ kilogauss, which renders the value of $I_m=775$ C.G.S. The wave-length used was $\lambda=530{\cdot}0\ \mu\mu$.

Cobalt-Tin.—The specimen was fused in the Metallurgical Laboratory of the Imperial College, London, and on analysis had the percentage composition—

Cobalt=93·2 per cent.
Tin= 6·8 per cent.

The mirror was prepared and tested as before, and with light of wave-length $\lambda=530{\cdot}0\ \mu\mu$ gave a negative maximum rotation of 14·3 minutes. The curve presented an abnormal appearance, insomuch that the transition portion is almost evanescent and the saturation point is almost reached before the bend occurs. The turn is very sharp and very small in gradient. The results are set out in Table XIV.

TABLE XIV.

Saturation curve. Cobalt-Tin. $\lambda=530{\cdot}0\ \mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
2·526	2·48	13·90	14·14
3·900	3·87	16·06	14·21
5·500	5·47	18·95	14·24
7·210	7·19	19·86	14·28
8·480	8·49	21·39	14·30
9·920	9·98	22·92	14·30
11·370	11·30	...	...

The rotation remained constant at 14·3 minutes for all higher values of the magnetic field. Rotation was negative.

From the above figures, when plotted in Fig. 14, the value of x is estimated as $x=14{\cdot}19$ kilogauss.

Therefore $I_m=1{,}129$ C.G.S.

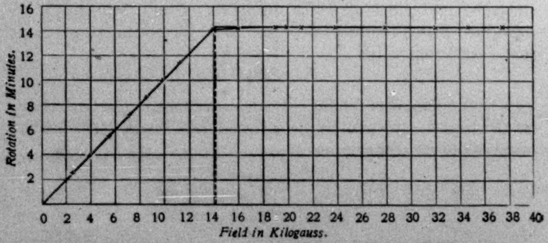


FIG. 14.—COBALT-TIN. SATURATION CURVE. ROTATION NEGATIVE.

A specimen of a cobalt-tin alloy which had been tested five years previously, and found to be magnetic, by Martin, was re-tested by Martin after three years and gave no rotation. The specimen was re-tested in the present work—*i.e.*, two years later—and again gave no result. It was then maintained at a temperature of 100 deg. Centigrade for two days and re-tested. Again a negative result was obtained. It was apparent, therefore, that in the five years some change had taken place in the alloy. The piece tested in Table XIV. is almost identical in composition with the original of Martin. This apparent ageing of the alloy we have been unable to explain up to date.

Cobalt-Antimony.—The material was prepared in London by the kindness of Prof. Merrett, of the Imperial College, who gave its percentage composition as

Cobalt=88 per cent.
Antimony=12 per cent.

The usual curve was obtained, with light of wave-length $\lambda = 530 \cdot 0\ \mu\mu$, and it presented no peculiarities. The maximum negative rotation was 18·72 minutes, and the value of x found was 14·50 kilogauss. The saturation value deduced from this gives

$$\underline{I_m = 1{,}155 \text{ C.G.S.}}$$

MINERALS, &c.—*Magnetite*, FeO, Fe_2O_3. *Magnetite* (1), (2) and (3).

General Remarks.—According to the curves for dispersion obtained by Loria and Martin in the case of magnetite, there is a negative rotation in the blue about wave-length $\lambda = 450 \cdot 0\ \mu\mu$ and a positive rotation in the orange-red about wave-length $\lambda = 600 \cdot 0\ \mu\mu$. In all cases where the saturation value of this substance was found two curves were taken using light of wave-lengths $\lambda = 450 \cdot 0\ \mu\mu$ (blue) and $\lambda = 600 \cdot 0\ \mu\mu$ (orange-red). In the first case the rotation was negative, and a slit width of 0·30 mm. was used on the monochromator. In the second case the rotation was negative and the slit width was decreased to 0·15 mm.

Magnetite.—(1) This specimen was taken from the end of an electrode and mounted in the usual way. Using light of wave-length $\lambda = 450 \cdot 0$, the maximum negative rotation obtained was 1·89 minutes, and using light of wave-length $\lambda = 600 \cdot 0\ \mu\mu$, the maximum positive wave-length was 3·83 minutes. Both curves gave exactly the same value for x—namely,

$$x=5{\cdot}190 \text{ kilogauss, which gives } I_m=\underline{413 \text{ C.G.S.}}$$

Magnetite.—(2) This specimen was kindly provided by Prof. Weiss, who stated that it had been artificially prepared. It was in an amorphous condition, and so had to be compressed strongly on to the cone prepared for insertion into the pole-piece of the magnet. In this way a mirror with a fairly good polish was obtained. Using the two wave-lengths as before, two good curves were traced. With wave-length $\lambda=450{\cdot}0\ \mu\mu$ the maximum negative rotation was 1·10 minutes, and using wave-length $\lambda=600{\cdot}0\ \mu\mu$ the maximum positive rotation was 2·25 minutes. Again both curves gave identical values for x—viz.,

$$x=5{\cdot}100, \text{ and therefore } I_m=\underline{406 \text{ C.G.S.}}$$

The two results are in good agreement.

Magnetite.—(3) This was a piece of natural mineral taken from the collection of Prof. H. du Bois. It was carefully mounted, as in the case of the earlier specimens, and the tests applied, using light of two wave-lengths as before. The curve obtained using light of wave-length $\lambda=450{\cdot}0\ \mu\mu$ gave a maximum negative rotation of 2·0 minutes, and the maximum positive rotation obtained with light of wave-length $\lambda=600{\cdot}0\ \mu\mu$ was 3·20 minutes.

The results of the two tests are set forth in Table XV. (*a*) and (*b*), and are traced in Figs. 15 and 16. The upper curve is the positive one.

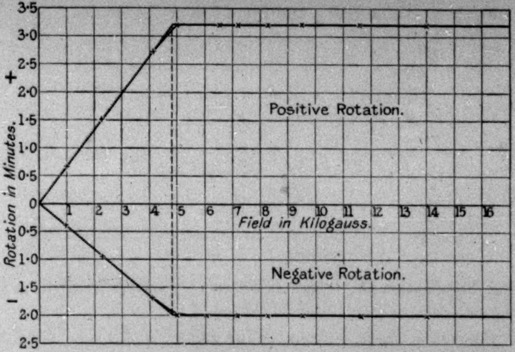
TABLE XV. (*a*) *and* (*b*).

Saturation curves. Magnetite (3). (*a*) $\lambda=450{\cdot}0\ \mu\mu$. (*b*) $\lambda=600{\cdot}0\ \mu\mu$.

Field in kilogauss.	$\lambda=450{\cdot}0\ \mu\mu$ rotation in minutes (negative).	$\lambda=600{\cdot}0\ \mu\mu$ rotation in minutes (positive).
1·00	0·40	0·675
2·30	0·95	1·50
4·10	1·70	2·75
4·95	2·00	3·20
6·55	2·00	3·20
7·20	2·00	3·20
8·25	2·00	3·20
9·50	2·00	3·20
11·60	2·00	3·20

For all higher values of the field the rotations were constant.

Figs. 15 and 16 both show the same value for x—namely, the value of $x=4{\cdot}850$ kilogauss, therefore $I_m=383$ C.G.S.



Figs. 15 and 16.—Magnetite (1). Saturation Curves.

Franklinite (ZnO, Fe_2O_3).—This specimen was also loaned by Prof. du Bois from his collection, and had the composition as expressed by the formula above. In order to select a suitable wave-length for the determination of the saturation curve, it was necessary to first determine the dispersion curve.

Table XVII.

Franklinite. Dispersion curve. Field = 27·34 kg.

Wave-length in μμ.	Rotation in minutes.	Wave-length in μμ.	Rotation in minutes.
442·5	−1·40	557·5	+0·78
457·4	−1·26	571·5	+0·92
474·0	−1·09	599·0	+0·98
492·5	−0·65	628·0	+1·05
515·5	+0·00	650·0	+1·12
531·5	+0·30	690·0	+1·19
546·1	+0·42	...	...

The procedure was just as usual, and the curve obtained very interesting. It commenced with a negative incline and passed through a zero value at $\lambda=515{\cdot}5\ \mu\mu$. After this it made a slight dip on the positive side, and then bent sharply round at

$\lambda=570{\cdot}0\ \mu\mu$. The curve then increased slowly towards the higher values of the wave-length. Throughout the whole experiment the field was saturated and maintained at 27·34 kilogauss. Table XVII. and Fig. 17 show the results.

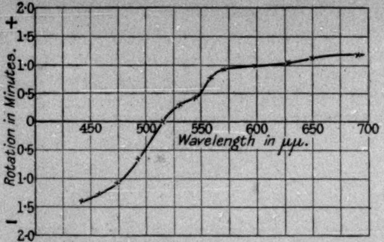


FIG. 17.—FRANKLINITE. DISPERSION CURVE. ROTATION NEGATIVE AND POSITIVE.

Saturation Curve.—From the foregoing curve it was thought best to select a wave-length of $\lambda=450{\cdot}0\ \mu\mu$ for the determination of the saturation curve. This entailed a monochromator slit width of 0·3 mm., and the rotation was negative throughout the whole curve. The results obtained are set out below in Table XVIII, and the curve shown in Fig. 18.

TABLE XVIII.

Saturation curve. Franklinite. $\lambda=450{\cdot}0\ \mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
0·8	0·5	3·7	1·19
1·1	0·7	4·9	1·19
1·7	1·1	5·8	1·19
2·85	1·19	7·5	1·19

Rotation is constant for all higher field values.

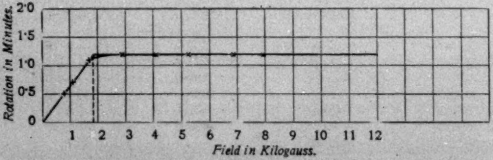


FIG. 18.—FRANKLINITE. SATURATION CURVE. ROTATION NEGATIVE.

Fig. 15 gives the value of x—

$$x=1\cdot870 \text{ kilogauss.}$$

Therefore $$I_m=149 \text{ C.G.S.}$$

Cupri-ferrite (CuO, Fe_2O_3).—This specimen was supplied as before by Prof. du Bois, and had the composition represented by the formula above. According to previous workers, it gave a positive rotation, and taking a monochromator slit width of 0·3 mm., light of wave-length $\lambda=477\cdot0\ \mu\mu$ was used for the curve. The usual series of values was obtained and Table XIX. gives the results.

TABLE XIX.

Cupri-ferrite. Saturation curve. $\lambda=477\cdot0\ \mu\mu$.

Field in kilogauss.	Rotation in minutes.	Field in kilogauss.	Rotation in minutes.
0·75	+0·4	4·1	+1·65
1·2	+0·6	5·15	+1·68
1·9	+0·95	7·50	+1·68
3·1	+1·55	...	...

Rotation is positive and is constant for all higher field values.

The curve quickly attains its maximum, and from it we find the value of

$$x=3\cdot350 \text{ kilogauss,}$$

which gives $$I_m=266 \text{ C.G.S.}$$

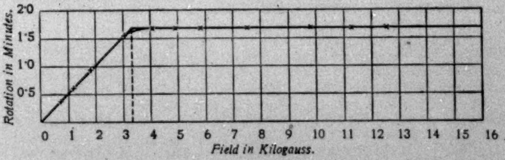


FIG. 19.—CUPRI-FERRITE. SATURATION CURVE. ROTATION POSITIVE.

In the final table are set out the whole of the values found by this method. The reliability of it is well proved in the first cases tested, and, as mentioned before, it is particularly good when only small quantities of the materials to be tested are available. In conclusion, I should like to thank most heartily Prof. H. du Bois, of Amsterdam and Berlin, for so kindly placing his apparatus and advice at my disposal for the work, Prof. Weiss, of Zurich, Dr. Gumlich, of Berlin, Asst. Prof. Merrett, of London, for so generously providing me with specimens, and Mr. J. Stephenson, A.R.C.Sc., B.Sc., for help during the writing of this Paper.

Material.	Maximum rotation in minutes.	Saturation value found in C.G.S.	Wave-length used in $\mu\mu$.	Otherwise found saturation value in C.G.S.	Source of material.	Remarks.
Iron (1)	−22·68	1,681	530·0	1,705	Gumlich ...	...
Iron (2)	−18·97	1,681	530·0	1,690	Weiss......	Fused in nitrogen
Iron (3)	−20·59 −25·13	1,675	546·1 590·1	1,702	Kolswa	Swedish.
Cobalt (1)	−20·58	1,400	530·0	1,412	Weiss.........	Fused in nitrogen.
Cobalt (2)	−21·00	1,395	530·0	1,406	Merrett	...
Nickel (1)	− 7·85	509	590·0	490	Weiss.........	...
Nickel (2)	− 8·05	513	530·0	...	Merrett	...
Ferro-Manganese (1)	−19·70	1,333	530·0	1,360	Gumlich ...	Fe=92·2% Mn=7·8%
Ferro-Manganese (2)	−13·60	873	530·0	885	Gumlich ...	Fe=89·6% Mn=10·4%
Iron-Cobalt-Antimony.	−23·66	1,607	530·0	...	Du Bois......	Fe=66% Co=30% Sb= 4%
Ferro-Cobalt	−27·64	1,850	530·0	...	Weiss.........	Fe_2Co.
Ferro-Nickel	−15·05	875	530·0	...	Weiss.........	Fe_2Ni.
Cobalt-Silicon	−20·52	1,094	530·0	...	London	Co=98% Si= 2%
Cobalt-Chromium ...	−13·72	880	530·0	...	London	Co=87% Cr=13%
Cobalt-Chromium-Nickel.	− 8·96	775	530·0	...	Tammann ...	Co=83% Cr=15% Ni= 2%
Cobalt-Tin	−14·30	1,129	530·0	...	Merrett	Co=93·2% Sn= 6·8%
Cobalt-Antimony ...	−14·50	1,155	530·0	...	Merrett	Co=88% Sb=12%
Magnetite (1)	−1·89 +3·83	413	450·0 600·0	...	Electrode ...	$FeO.Fe_2O_3$.
Magnetite (2)	−1·10 +2·25	406	450·0 600·0	...	Weiss.........	As above.
Magnetite (3)	−2·00 +3·20	383	450·0 600·0	...	Du Bois......	As above.
Franklinite	−1·19	149	450·0	...	Du Bois......	$ZnO.Fe_2O_3$.
Cupri-Ferrite	+1·68	266	477·0	...	Du Bois......	$CuO.Fe_2O_3$.

ABSTRACT.

The Paper describes work carried out in the laboratory of Prof. Du Bois on the relation between the intensity of magnetisation of various ferro-magnetic materials and the rotation of the plane of polarisation of plane polarised light reflected from a polished surface of the material. The specimens, in the form of circular discs 5 mm. in diameter and 0·5 mm. thick, were soldered to one of the pole pieces of a large electromagnet. Through an aperture in the other pole, monochromatic light, polarised in two nearly coincident planes by means of a Lippich polariser, was incident almost normally on the polished surface of the specimen. The reflected beam passed through an analyser, the rotation of which could be measured, by means of an auxiliary optical system, to a high degree of accuracy.

In the first part of the Paper results are given for a number of materials of known magnetic properties in order to establish the validity of the method, due to Du Bois, of obtaining the value of the saturation intensity from the curve connecting field strength with rotation. The method is then applied to materials of unknown properties. The variation of the Kerr constant with the wave-length of the light was also determined for a number of substances.

DISCUSSION.

Mr. D. Owen asked if the Paper, in addition to establishing the view that the rotation was proportional to the intensity of magnetisation, brought forward any new results that were useful to theory. For instance, he recollected a Paper by B. Hopkinson, in which it was found possible to obtain an intensity of magnetisation greater than that obtainable with pure iron, and he noticed that in the present Paper there was an alloy of iron and cobalt which exceeded pure iron in saturation intensity. Was it now taken for granted that it was possible in an alloy to obtain magnetisations greater than that of either component, or were the results in the Paper somewhat novel in this respect ?

Dr. Smith said that saturation intensities of magnetisation were of importance for various reasons. For instance, as Weiss had shown, it was of theoretical interest to know their temperature variations. Again, in the case of a " mechanical " mixture of two substances, the saturation intensity would be related in a simple manner to those of the constituents. There was much evidence that annealed steels are mixtures of iron and iron carbide. Hadfield and Hopkinson, in the Paper which had been mentioned, had shown that the saturation intensities of magnetisation of such steels were compatible with this view, and had deduced a value for the saturation intensity of the carbide. It would have been of interest if Dr. Barker had determined the latter directly, as had been his intention, before the work was interrupted. Alloys were not, however, always simple mixtures. The steel examined by Hopkinson and Hadfield to which Mr. Owen had referred, was a nickel steel, and its total magnetisation was greater than the sum of those which its constituents would have exhibited separately, although it was not as great as that of pure iron. The reason probably was that nickel-iron alloys form solid solutions (mixed-crystals) instead of mechanical mixtures. The same is probably true of cobalt-iron alloys, although in this case a material can be obtained (possibly a compound) whose saturation magnetisation is actually greater than that of pure iron. It had been proposed to use this alloy commercially. Other solid solutions were known in some of which the magnetisation was greater, and in others less, than that which mechanical mixtures of the components would have exhibited.

Dr. P. E. Shaw (communicated remarks): The peculiar features of the interesting method employed by Dr. Barker are the smallness of the specimens and the largeness of the field used. In order to obtain rotation effects of fair magnitude it would seem to be necessary to deal exclusively with magnetic materials of ferromagnetic strength, and even then to employ great fields. This limitation in the scope of the method is, however, counterbalanced by the great advantage of being able to experiment on very small quantities of the magnetic substance. It would thus seem easy to conduct useful experiments on the change in magnetic quantity of the small specimens when the conditions as to temperature, annealing and static treatment are varied, all of which tests might be difficult with large specimens. Throughout the Paper the values of λ are given in terms of $\mu\mu$. The unit here meant is one-millionth of a millimetre, which is now, according to recent changes in nomenclature, styled $m\mu$. The Society recently sanctioned this change.

II. *The Influence of the Time Element on the Resistance of a Solid Rectifying Contact.*

By DAVID OWEN, *B.A., B.Sc., Birkbeck College and Borough Polytechnic Institute, London.*

RECEIVED OCTOBER 2, 1916.

THE properties of a rectifying contact may be studied by examining the curve connecting its resistance with the value of the steady applied voltage. In a previous Paper* some examples of such curves, or resistance characteristics, were given, and from its properties an expression was found for the sensitiveness of the contact treated as a rectifier of high-frequency oscillatory currents. It is, however, an obvious experimental fact that the resistance of a contact depends appreciably on the time of application of the voltage. Considerable definiteness is secured by applying the testing voltage for a small fraction of a second. The form of the characteristic determined in this manner is, moreover, explicable in terms of simple thermo-electric considerations. Experiments since carried out with the object of more fully determining the influence of the time element form the subject-matter of the present Paper.

The results have a bearing in the first place on the calculation, above alluded to, of the sensibility of a wireless receiving circuit including the contact. They have a further interest in their bearing on the thermo-electric theory of the contact; in connection with which arises the question of the location to be ascribed to the thermal effects which (on this theory) accompany and give rise to the observed variations of electric resistance.

The conclusion is reached that there is present at the interface of the contact a stratum of molecular thickness in which resides a considerable proportion of the total resistance of the contact; and that it is only the changes occurring within this stratum that are capable of causing rectification at frequencies such as are used in wireless telegraphy.

The subject of the time-lag of resistance at a contact has been previously examined by Pierce,† who, in experiments made with alternating currents of 80 ∿ was unable to detect the existence of any lag. From the observations recorded below it follows that such a lag is necessarily present, though not to an extent which would be probably appreciable in cathode ray oscillograms taken under the conditions of the experiments quoted.

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* "Proc. Ph. Soc., London," Vol. XXVIII., p. 173, June, 1916.

† "Phys. Rev.," 1907.

The Method of Measurement.

The Wheatstone Bridge was again employed. It is required to measure the resistance of a contact at a known time, which may be a small fraction of a second, after completing the battery circuit. When the time is a second, or more, the ordinary method of obtaining a balance can, of course, be employed. For the shorter times a ballistic method was substituted. The galvanometer key being closed, the battery circuit is closed for the period determined upon; the process being repeated until, by adjustment of the variable arm, the galvanometer throw is reduced to zero.

Two methods of making the battery circuit were used. For the shortest periods—from $\frac{1}{40000}$ second to $\frac{1}{200}$ second—the circuit was closed by allowing a polished metal ball or ring suspended by a fine wire to strike the filed and polished surface of a heavy piece of steel plate. The duration of the collision depends on the ball or ring used, and to a slight extent on the distance of its withdrawal from the plate before release. This time was independently ascertained by measuring the charge communicated, during the collision, to a mica condenser in series with a resistance of known and suitable value. The data thus obtained are given below :—

TABLE I.—*Periods of Collision.*

No.	—	—
1	Steel bicycle ball, diameter 0·71 cm.	0·000025 second
2	Brass sphere, diameter 2·54 cm.	0·00012 ,,
3	Brass sphere, diameter 5·1 cm.	0·00026 ,,
4	Brass ring, diameter 5·0 cm.	0·00036 ,,
5	Brass ring, diameter 10·0 cm.	0·00082 ,,
6	Brass sphere, diameter 21 7 cm.	0·00112 ,,
7	Manganin wire ring, diameter 10 cm.	0·0031 ,,

At the shortest periods the difficulty naturally arises of maintaining the sensibility of the bridge. A Broca galvanometer of 1,000 ohms was employed, and the bridge was arranged to give as far as possible maximum sensibility. The sensibility was then increased by repeating the collision a number of times in rapid succession. It was found that this could be done satisfactorily by hand. The ball was first soldered to a suitably stiff wire which was then insulated with silk and used as a handle. It is possible in case of the bicycle ball to strike the steel plate some five or six times per second. Since the period of the galvanometer needle was about 10 seconds, one could use with effect some 15 strokes, thereby gaining an increase in the accuracy of measurement in nearly that proportion. In this way it was found that a resistance of 1,200 ohms was measurable to within one ohm when the period of closure of the battery circuit was 0·000025 second, the voltage applied to the bridge being 2 volts. Since the interval between successive strokes is about one-fifth of a second, a time very large compared with the duration of one contact, there is no doubt that the resistance which the operation furnishes is that corresponding to a single stroke.

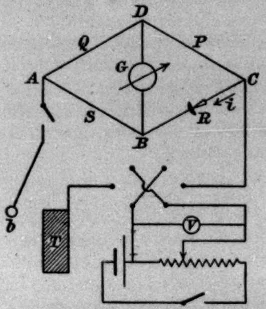

Fig. 1.—Diagram of Bridge for Ballistic Test.
The battery circuit is momentarily completed by allowing the ball or ring *b* to collide with the metal plate *T*.

This collisional method becomes inconvenient when the time of contact required is greater than about 0·005 second. For times ranging between 0·005 second and 0·5 second a revolving disc of wood of diameter 14 cm. was used, into the periphery of which were let several sectors of brass of differing size. Metallic connection could be made between any one of these and the spindle, which was of gun-metal. Circuit was completed through brass brushes. The disc could be driven uniformly at any speed between 50 and 150 revs. per min. It is easy by means of a simple key to confine observations to the effect of a single closure of the circuit. The electrical connections of the bridge are shown in the diagram of Fig. 1.

Theory of the Ballistic Bridge.—When one of the four arms of a Wheatstone Bridge contains a resistance which varies during the period of application of the voltage, it is evidently impossible to find a balance in the sense that no current whatever flows in the galvanometer branch. It is possible, however, to adjust the variable arm so that the total quantity of electricity passing through the galvanometer is zero.

Let R denote the instantaneous resistance of the contact under test; P, Q and S, the resistances of the remaining arms; and G the resistance of the galvanometer. The resistance of the battery branch may be kept negligibly small; this results in some simplification in the theory. Let

V=voltage across bridge (A to C),
v=instantaneous voltage across R (B to C),
i=instantaneous current through R,
t=time from instant of closure of battery circuit.
τ=total period of closure of circuit,
$\bar{v}$ and $\bar{i}$ denote mean values of v and i over the time t.

Applying Kirchhoff's equations, and observing the condition that the integral flow of electricity through the galvanometer is zero, we have at balance

$$\frac{P}{Q}=\frac{\int Ridt}{\int Sidt}=\frac{\int vdt}{S\int idt}=\frac{\bar{v}}{S\bar{i}}\quad\ldots;$$

therefore
$$\frac{P}{Q}\cdot S=\frac{\bar{v}}{\bar{i}}.\qquad\ldots\ldots\ldots\quad(1)$$

Again
$$\bar{v}=V-S\bar{i}=V\cdot\frac{P}{P+Q};\qquad\ldots\ldots\quad(2)$$

whence
$$\bar{i}=\frac{VQ}{S(P+Q)}.\qquad\ldots\ldots\ldots\quad(3)$$

It will be seen that $\bar{v}$ and $\bar{i}$ are at once calculable in terms of V and the resistances of the three known arms.

From a series of measurements with different values of τ a curve can be drawn connecting $\bar{i}$ and t. From this may be determined the value of the current at the end of any time within the range employed. For since $\bar{i}=\frac{1}{t}\int_0^t idt$, it follows by differentiation that

$$i_t=\bar{i}+t\left(\frac{d\bar{i}}{dt}\right)_t\qquad\ldots\ldots\ldots\quad(4)$$

The simultaneous values of v and R at any moment can thence be calculated. The expression found for the former is

$$v=V\cdot\frac{PS+PQ+G(P+Q)}{PQ+(S+G)(P+Q)}-i\cdot\frac{PQS+GS(P+Q)}{PQ+(G+S)(P+Q)} \quad . \quad . \quad (5)$$

Having thus calculated v, the value of the resistance at the same instant may be found, since

$$R=v/i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

It will be observed that the expressions for instantaneous values involve the galvanometer resistance.

It was found that the curves thereby obtained connecting R and v were of the same general trend as those connecting $\frac{P}{Q}\cdot S$ and $\bar{v}\left(=V\cdot\frac{P}{P+Q}\right)$. The latter quantities were, therefore, plotted to represent R and v respectively in the characteristic curves shown in the Paper.

The balance is independent of the inductances in the bridge network, since the initial and final values of all the currents are zero.

Statement of Results of Observation.

Experiments were conducted on zincite, galena, and carborundum; the other element of the contact being platinum in each case. Two sets of determinations were made, one connecting R and v (τ constant); the other connecting R and t (v constant). The results are expressed in the accompanying curves and tables.

Attention was concentrated mainly on zincite, owing to the reliable nature of the contacts which this material furnishes. Many points of contact were tried. Two sets of resistance characteristics, obtained from two of these points, are given in Figs. 2 and 3; they include the main features found in all the tests. Parts of the curves are omitted where excessive closeness might cause confusion. It may be seen that—

(1) For current across the contact from zincite to platinum the resistances increases with time, the reverse being true when the current is in the reverse direction. This result holds at all values of the voltage. The changes are more pronounced for the first mentioned direction of current. The curves all pass through the same point at zero voltage.

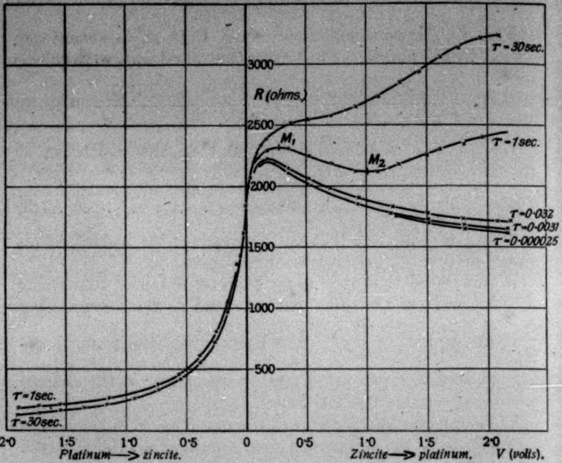

Fig. 2.—Resistance Characteristics for various periods of Application of Voltage. (Zincite-platinum.)

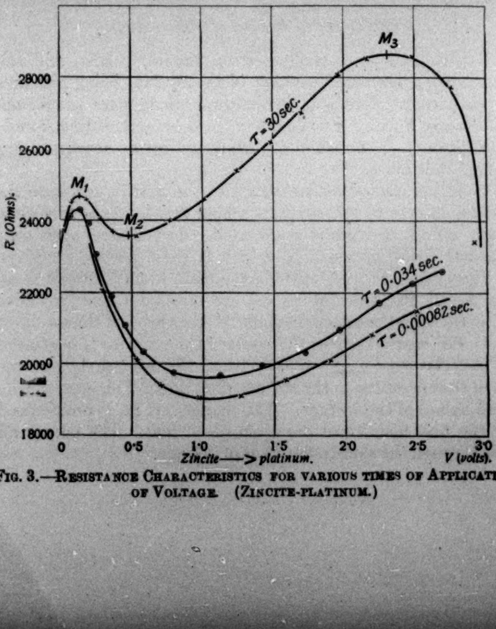

Fig. 3.—Resistance Characteristics for various times of Application of Voltage. (Zincite-platinum.)

(2) The effect in the neighbourhood of zero voltage is a slight increase in the slope, or voltage-coefficient, with increase in the time of application of the voltage.

(3) At higher voltages, for current from zincite to platinum (across the contact) the increase of resistance with time may be so great as to produce a minimum point M_2 in the characteristic. This effect is in some cases (*see* Fig. 2) great enough to obliterate the first maximum, which is characteristic of all the curves when τ is small.

(4) When the voltage is increased to a value between 2 and 3 volts, there is a steep fall in the characteristic. At the larger values of τ this is preceded by a second point of maximum, which is more conspicuous for some contacts than for others. (Cf. M_3 in Figs. 2 and 3.)

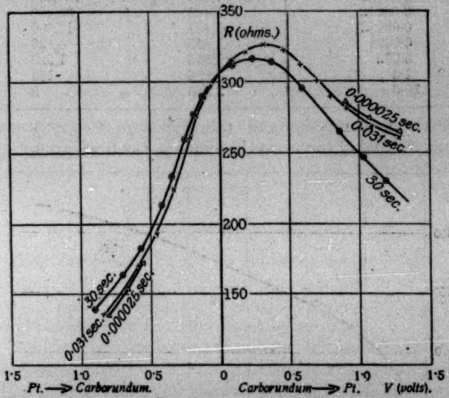

FIG. 4.—RESISTANCE CHARACTERISTICS FOR VARIOUS PERIODS OF APPLICATION OF VOLTAGE. (CARBORUNDUM-PLATINUM.)

Experiments on contacts of *galena* show at low voltages effects of the same kind as those of zincite; that is, the slope of the characteristic at zero voltage is accentuated by increasing the duration of flow of the current.

In the case of *carborundum* the time effect is contrary in sign to that in the two preceding cases. As shown in Fig. 4, the resistance of a carborundum contact decreases as the current is prolonged, when the direction is from carborundum to platinum across the contact; the effect being as before reversed when the current is reversed.

The time-effect may be examined from another point of view by determining the resistance of a contact for a series of gradually increasing periods of closure of the battery circuit, the applied voltage being the same in each case. The data shown in Table II. were found in the case of a zincite contact. Applied voltage=2·08 volts; resistance at zero voltage, R_0 =1,311 ohms; voltage coefficient=4·4; current from zincite to platinum across the contact.

TABLE II.—*Variation of Resistance with Duration of Current.*

Time of application of voltage (seconds).	Resistance (ohms.)	*dR/dt* (ohms/sec.)
0·000025	1,453	140,000
0·00012	1,463	36,400
0·00082	1,477	13,100
0·0031	1,495	4,820
0·032	1,545	1,220
1·0	1,805	163
30·0	2,245	8

The third column shows the rate of change of resistance with time at the end of the period specified in the first column.

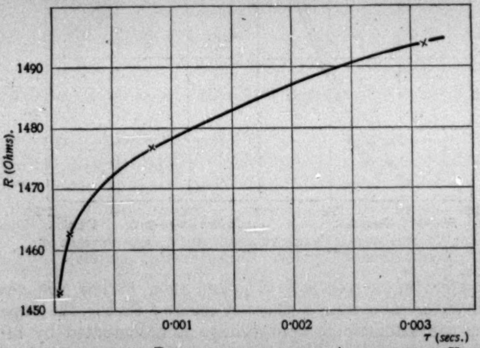

FIG. 5.—VARIATION OF RESISTANCE WITH TIME OF APPLICATION OF VOLTAGE (ZINCITE-PLATINUM CONTACT. R_0=1311 OHMS; v=2·08 VOLTS.)

These observations cannot be represented adequately on a single diagram with one time scale. The curve of Fig. 5 includes the first four points.

As the changes of resistance observed within the range of the part of the curve shown are not great, it may be well to state that great care was taken to avoid errors arising from the variations, steady or erratic, which go on more or less at all contacts when left to themselves. For example, measurements at $\tau=0{\cdot}00082$ second and at $\tau=0{\cdot}00012$ second would alternate with one another until certainty as to the difference of resistance corresponding to these two periods was secured.

From the above table and figure we gather

(5) That the rate of variation of resistance becomes continuously greater as τ becomes less.

Nevertheless, if the time of flow of current be divided into steps increasing in geometrical progression, it is found that, up to a point, the absolute change of resistance in any step becomes greater as we progress along the series. This is shown in Table III., which is deduced from Fig. 5.

TABLE III.

Interval (seconds).	Change of resistance (ohms).
=0·000025 to =0·00025	1,467·5 − 1,453 = 14·5
=0·00025 to =0·0025	1,492·0 − 1,467 = 35·0
=0·0025 to =0·025	1,537·0 − 1,492 = 45·0
=0·025 to =0·25	1,660·0 − 1,537 = 123·0

Interpretation of Results.

The above data enable a judgment to be formed of the relation between the characteristic actually described by a contact under an oscillatory voltage of, say, a million frequency, and that obtained in a slow point by point test.

The time of application of voltage has been varied from 30 seconds to a little under one millionth of that time. A further reduction to one-tenth the lower limit would have yielded a period of the order of 10^{-6} second. The degree of extrapolation involved appears sufficiently small to warrant the conclusion that the characteristic at $\tau=10^{-6}$ second would not materially differ from that obtained at $\tau=0{\cdot}001$ second, or even 0·01 second; though referred to that found at $\tau=1$ second or more the correction required would be very appreciable, especially at the higher voltages.

The sensitiveness of the contact, used as a detector of high-frequency oscillations, and in the absence of a polarising voltage, would in the case of zincite or galena be probably less by some 10 per cent. or 20 per cent. than that deduced from a long-period characteristic; in the case of carborundum it would be greater by approximately the same percentage.

The bearing of the experimental results on the thermo-electric theory of the contact will now be examined. According to this theory the changes of resistance with voltage are not instantaneous, but are a consequence of the thermo-E.M.F. and the temperature variation of resistance that accompany heat development at the contact. The rate of variation actually observed is very great. To quote one case (*see* Table II.), the resistance of a zincite contact under a voltage of 2·08 volts was 2,245 ohms at the end of 30 seconds; at an instant 0·000025 second after application of the voltage it was 1,453 ohms, the rate of variation then being 140,000 ohms per second. The limiting resistance was 1,311 ohms. It appears not unreasonable to suppose that the outstanding difference of 142 ohms occurred in the first 40,000th of a second; and that the whole difference between the resistance at zero voltage and that found at any other voltage is due to and follows the temperature variation at the contact.

As already pointed out, the characteristic at $\tau=0{\cdot}000025$ second does not very materially differ from that obtained at $\tau=0{\cdot}01$ second, whereas the latter differs considerably from the characteristic at $\tau=1$ second. Again, it appears (*see* Fig. 4) that an initial increase of resistance may be followed by a decrease as the voltage is sustained, or vice versa. These facts point to the necessity of modifying the simple view that the Joulean and the thermo-electric disengagements of heat are located in precisely the same volume of the crystal.

They become intelligible if we suppose two regions of heat development. The question of the location of the resistance at a contact was dealt with (from an altogether different point of view) in the preceding Paper (*loc. cit.*). Two views were discussed. According to the first the resistance may be located within the high-resistance element of the contact, in a volume whose linear dimensions in every direction are of the order of those of the interface $\left(R \doteqdot \frac{0{\cdot}8\rho r_0}{\pi r_0^2}\right)$. According to the second view the resistance may be assigned to a stratum of molecular thickness, but of considerable resistance, at the interface of the two elements of the contact.

These views are not mutually exclusive, and the data at present under discussion give grounds for the conclusion that the resistance is made up of the two in series, as illustrated in the diagram of Fig. 6.

The Peltier heat would be developed solely in the second region. Joulean heat would be set free in both regions, as also would the Thomson heat. Joulean heat is always positive. The Thomson heat, like the Peltier heat, may be either positive or negative; but in contrast to the Peltier heating effect, its sign is (for small currents) unaffected by the direction of the current.

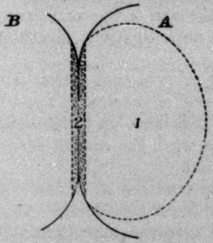

FIG. 6.—DIAGRAM ILLUSTRATING VIEW OF TWO REGIONS OF HEAT DEVELOPMENT AT A CONTACT.

A represents the high-resistance element; *B* the low-resistance element of the pair. Region 1 has linear dimensions of the order of those of the interface; region 2 is a stratum of molecular thickness

Extremely rapid change of temperature is regarded as associated with heat liberated (or absorbed) in region 2; and slow change with thermal effects occurring in region 1. There are thus two "time-constants," one, proper to the molecular stratum, being very small compared with the other.

The Thomson heat may possibly account for the difference in the two types of crystal, one represented by zincite, the other by carborundum (*see* Figs. 2 and 4). If the thermo-E.M.F. be represented by a parabolic formula (the experiments of Bidwell* appear to point to this law for oxides, sulphides, &c.), then writing

$$e=a(T-T_0)+c(T^2-T_0^2) \quad . \quad . \quad . \quad . \quad . \quad (7)$$

in place of the simple linear formula used in the previous Paper, we find for the slope of the characteristic at zero voltage the modified expression

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* "Phys. Rev.," 1914.

$$\left(\frac{dR}{dv}\right)_0=\frac{-R_0\frac{P_0}{T_0}\left(1+bT_0+\frac{cP_0T_0}{KR_0}\right)}{KR_0}, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

in which P_0=Peltier coefficient at the absolute temperature T_0 of the surroundings; b=temperature coefficient of resistance; K=dissipation constant.

If the Thomson heat develops slowly, then the question of increase or decrease with time of the slope at zero voltage will be determined by the sign of the coefficient c. A zincite contact behaves as if c were negative; a carborundum contact as if c were positive.

A Thomson absorption of heat (accompanied by a negative temperature-coefficient of resistance) appears also a probable cause of the rise of resistance up to the second maximum (*see* Fig. 3) which occurs when the current is sustained for a comparatively long period.

The characteristic obtained at a zincite contact when the voltage is sustained for a considerable period is conspicuous by the presence of a second maximum. The curve rapidly descends past this point, and once this has been permitted to happen the limiting resistance of the contact is found to be only a small fraction of the initial value. This drop, which is permanent, is to be assigned to softening due to rise of temperature; this leading in its turn to increase of area and consequent reduction in electric resistance. The effect here outlined is doubtless to be identified with the "breakdown" found in practice when a contact has been subjected to an excessive voltage.

The Values of the Time Constants.—Accepting thermoelectric action as the true basis of the rectifying property, a time lag of resistance must inevitably occur; but this, if it is not to nullify the rectifying action, must be of comparatively small account even at the rapidity of change involved at electric wave frequencies. This can only be the case if an appreciable portion of the resistance of the contact is associated with a time-constant small compared with the period of the impressed oscillations.

It is possible to form estimates of the time-constants corresponding to the regions 1 and 2 of Fig. 6; the dimensions of the contact, the thermal conductivity and the specific heat of the material entering as important factors. The heat dissipation may be attributed entirely to conduction to neighbouring portions of the crystal—this restriction will tend to yield a maximum estimate. Two problems may first be considered in the abstract.

Problem I.—Suppose heat to be developed at a constant rate within a small sphere in the interior of an infinite uniform solid mass. For simplicity assume the temperature to be constant at any instant at all points within the sphere. What is the time-constant which defines the rate of rise of temperature of the boundary of the sphere?

Let θ_0 denote the initial temperature, θ the final temperature of the sphere, k the thermal conductivity, d the density, s the specific heat, H the rate of supply of heat, r_0 the radius of the sphere, and T the time-constant required.
It is easy to show that

$$\theta-\theta_0=\frac{H}{4\pi k r_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

The temperature change occurring in accordance with a simple exponential law, we have

$$T=\frac{\theta-\theta_0}{(d\theta/dt)_0}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Also $$\left(\frac{d\theta}{dt}\right)_0=\frac{\text{rate of supply of heat}}{\text{thermal capacity of sphere}}=\frac{H}{\frac{4}{3}\pi r_0^3 ds}. \quad . \quad (11)$$

Combining the above equations,

$$T=\frac{r_0^2 ds}{3k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

Now in the case of zincite, &c., we may take $ds \doteqdot 0{\cdot}5$, and $k \doteqdot 0{\cdot}005$, whence

$$T \doteqdot 33r_0^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Hence, if $T<10^{-7}$ second, then $r_0<0{\cdot}5\times10^{-4}$ cm.

The second problem is intended to relate to a stratum of molecular thickness, such as a circular disc, enclosed within an infinite solid. The difficulty of dealing with the edge effect may be avoided without materially affecting the calculation if we substitute a spherical shell of the same area. We may then state the problem thus:—

Problem II.—Suppose a spherical shell of molecular thickness forming the boundary of a spherical hollow in the interior of an infinite solid; and that within the material of the shell heat is *developed at a constant rate. Assuming equality of temperature within the shell at any instant, what is the value of its thermal time-constant?*

Using the same symbols as before, we have

$$\theta-\theta_0=\frac{H}{4\pi k r_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14)$$

and $$\frac{\theta-\theta_0}{T}=\frac{\text{rate of supply of heat}}{\text{thermal capacity of shell}}=\frac{H}{4\pi r_0^2 ds \,.\, 10^{-8}}, \quad . \qquad (15)$$

whence $$T=\frac{r_0 ds}{k} \,.\, 10^{-8} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (16)$$

Taking $ds \doteqdot 0{\cdot}5$, $k \doteqdot 0{\cdot}005$, we find $T \doteqdot r_0 \,.\, 10^{-6}$ second. Hence if $T < 10^{-7}$ second, then $r_0 < 0{\cdot}1$ cm.

We may identify the conditions of the above problems with those occurring in regions 1 and 2 respectively of a rectifying contact. The assumption of infinite conductivity within the small sphere and in the stratum has the effect of reducing the value of T calculated, and hence of increasing the value fo r_0 corresponding to a selected value of T. The attendant error only enhances the deductions drawn, namely, that the variations of temperature and resistance at a contact can only follow the voltage without serious diminution at electric wave frequency—

(*a*) In region 1 if the mean diameter of the area of contact $< 0{\cdot}5 \times 10^{-4}$ cm.;

(*b*) In region 2 if the mean diameter $< 0{\cdot}1$ cm.

Now in actual practice the mean diameter of the area of a good rectifying contact is as a rule far greater than 10^{-4} cm. The conclusion is that the only variation in resistance that can be effective in causing rectification at very high frequency must have its origin in a stratum of molecular thickness at the actual interface of the contact. What the composition of the stratum is there are not at present sufficient data to determine. It is required of it that it should be very thin—indeed of molecular thickness; that it should have a resistivity far higher than that of the material of the contact treated in bulk; and that it should have a negative temperature-coefficient of resistance.

This conclusion accords both with the experimental results and with the calculated maximum rate at which it is possible for a defined volume of material to vary in temperature when subjected to a constant source of heat. That the rate of supply of electrical energy which is found to operate the receiving circuit of a wireless station is adequate to produce the required thermal effects at the rectifying contact may be shown by simple calculation.

SUMMARY.

The resistance at a solid rectifying contact, and consequently the exact shape of the resistance characteristic, depends upon the time for which the testing current is allowed to flow. A series of characteristics are given corresponding to durations of contact extending over the range one forty-thousandth of a second to 30 seconds.

The following conclusions are drawn :—

1. That the variation of resistance with voltage may be attributed entirely to thermal effects.
2. That the characteristic obtained by applying the testing voltage for one hundredth of a second is, at moderate voltages, materially the same as that which would be found at the expiry of a period of the order of a millionth of a second.
3. That the sensibility of a wireless receiving circuit (in which the rectifying contact is used) does not differ very appreciably from that deduced from a slow period characteristic.
4. That an important fraction of the contact-resistance resides in a stratum of molecular thickness at the interface of the two elements of the contact ; and that it is in this region alone that rectifying action at very high frequencies is effected.

DISCUSSION.

Prof. G. W. O. **Howe** thought the experimental method of Mr. Owen was the correct way to tackle the problem of high frequency phenomena, and was preferable to taking oscillograph records at low commercial frequencies, and attempting to deduce from them what will happen at very much higher frequencies. What is the limit to the number of times per second that the ball may be struck for the results still to apply to one impact only ? From one point of view the results were disappointing, inasmuch as it had been necessary to introduce a mysterious stratum of molecular thickness and the problem arises as to what these layers consist of ; what are their properties, and why is the thermal effect largely confined to them ?

Dr. H. S. **Allen** asked if any attempt had been made to remove surface films from the contacts. Prof. Millikan had recently obtained very simple results in photo-electricity by working in a high vacuum, so as to remove all films of moisture. Could this not be done in Mr. Owen's experiments ?

Dr. **Eccles** said that the subject of contact resistance was of wide importance, apart from its application to electric wave detectors. It had been investigated by Wheatstone and Heaviside in other connections—microphones, for example. It was important also in connection with the contacts at dynamo brushes. There had always been a desire in wireless telegraphy circles for high-frequency characteristic curves. Doubtless. if oscillograph records could be obtained for frequencies of 10^{-6} second, they would be very like what Mr. Owen gets by his point to point method. He would have been glad if the author had tried similar contacts—*e.g.*, two carbons, two galenas, or even two metals. In these cases there would be no thermo-electric phenomena to confuse the issue. He was not convinced on the question of the molecular layer. The author states that "in actual practice the mean diameter of the area of a good rectifying contact is, as a rule, far greater than 10^{-4} cm." Now, if one pictures a piece of metal resting on the zigzag surface of a crystal, it is obvious that actual contact will only occur at points and ridges, and it is surely impossible for anyone to assert what the order of the area of contact is. He preferred to think that the effects were due to the heating of the surface layers of the stuff, due to concentration of the current flow into the narrow regions where actual contact exists. Usually, the harder the crystal the better the rectification obtained, due possibly to the greater concentration of current in the smaller and sharper contact points. He thought perhaps too much stress was laid on the high-frequency characteristic. When an oscillation passes through the contact of a crystal touching a metal plate it is the after effects of the heat developed which produce the thermo-electric currents that affect the telephone. Therefore, while of scientific interest, the high-frequency characteristics are not very essential from the point of view of wireless telegraphy.

Mr. F. E. Smith thought Dr. Allen's suggestion a valuable one. About 14 years ago a Paper was read to the Society showing that films of liquid remained attached to solids even up to temperatures of 300°C.

The Author, in reply to Prof. Howe, thought there was no mystery in the contact layer. We cannot assume complete continuity of material at a contact as in a weld. This is evidenced by the fact that no appreciable force is required to separate the elements, though they have been pressed together with a force involving a considerable minimum area of contact. With respect to the repeated impacts (used in order to raise the sensibility of the bridge), the time between the successive impacts was never less than about 8,000 times the duration of a single contact. The point raised by Dr. Allen, and emphasised by the chairman, deserved investigation. Dr. Eccles raises two important points. In order to avoid the necessity of assuming a molecular stratum, he suggests the view of multiple contacts. This, if carried out to the requisite minute subdivision of the total contact area, will certainly effect the required reduction of the thermal time-constant. I have taken the opportunity of submitting the matter to calculation. I find that the time-constant will be reduced to 10^{-7} sec., provided the number of effective contacts, all of equal area, is not less than 25,000, and this number must, on grounds of probability, be regarded as only a small proportion of the total number of contacts under the total area, on the view under discussion. It seems a case of choosing between a molecular layer and an ark resting on a hundred thousand Ararats, the size of each peaklet not far removed from molecular dimensions. Dr. Eccles also asserts the view that the electric discharge which actuates the telephone in a wireless receiving circuit is due to the *after effect* of the heat accumulated at the contact; but this is to neglect altogether the rectified current that traverses the contact during the actual period of impress of the electric oscillations, and this is amenable to calculation (*see* Paper in Vol. XXVIII., p. 196). I find that the impulsive flow due to the rectified current would be sufficient to produce an audible effect in the telephone provided the mean amplitude of the wave of P.D. across the contact during the passage of a train of oscillations is somewhere between one hundredth and one-tenth of a volt. Now at such low voltages the reversible Peltier heat (on which rectification depends) in the case of crystals like zincite exceeds the irreversible Joulean heat (on which the after effect depends). Curious as it may at first sight appear, the power expended in the alternate Peltier heating and cooling greatly exceeds, at low voltages, the total electrical power supplied to the rectifying sector by the incident oscillations. There is thus reason for regarding the signal in the telephone as due mainly to the rectified current. In reference to phenomena at a contact between similar materials, I have at present under consideration the investigation of the effect of pressure at such contacts.

III.—*On Diffusion in Liquids.* *By* BASIL W. CLACK, *M.Sc. (Lond.), Lecturer in Physics at Birkbeck College.*

RECEIVED SEPTEMBER 28, 1916.

§ 1. *Introduction.*

IT has recently been pointed out by Griffiths [" A Recalculation of some Work on Diffusion," Proc. Phys. Soc., Lond., XXVIII., p. 255, 1916] that a correction is necessary to the theory of the method which has been used by the author for studying the diffusion of salts through water. [*See* Clack: (1) " On the Coefficient of Diffusion," Proc. Phys. Soc., XXI., p. 863, 1908; (2) " On the Temperature Coefficient of Diffusion," Proc. Phys. Soc., XXIV., p. 40, 1911; (3) " On the Coefficient of Diffusion in Dilute Solutions," Proc. Phys. Soc., XXVII., p. 56, 1914.] In these Papers experiments are described in which diffusion has been studied by means of special flasks, filled with the solution under investigation, and suspended in a large bath of water at known constant temperatures. The design of the flasks is such as to maintain the concentration of the solution constant at each end of the vertical tubes which are fitted into the flask, so that diffusion takes place in the steady state, that is to say, the concentration of the solution at any point in the tube remains constant while the measurements are made.

The decrease in volume of the solution in the flask as diffusion reduces its concentration, causes a current of liquid to flow down the tube, so that as the salt diffuses upwards it meets a current of liquid which continually passes downwards, and which is due to the decrease in volume referred to. In the Papers mentioned above, reference was made to the velocity of this current of solution; but the matter is complicated by the fact that simultaneously with the movement of the water in the solution the dissolved salt is diffusing, and therefore may have quite another velocity, and on this account it is necessary to confine attention to the velocity of the water component of the solution.

§ 2. *Objects.*

The objects with which this communication is made, are two:—

1. To amend the theory in the direction indicated; to present the recalculated values of the coefficient of diffusion at different temperatures and at various concentrations; and to

evaluate the temperature-coefficient of diffusion in the case of certain solutions.

2. To deduce an expression for the coefficient of diffusion, K_n, at any definite concentration, n, which is now rendered possible by the simpler form of the corrected formulæ deduced in the next section; and to find the numerical value of this quantity for specified concentrations of solutions of certain salts.

§ 3. *Theory.*

The theory previously published in the first Paper mentioned in § 1 must be amended as follows :—

Let L = total length of the diffusion tube.
N = constant concentration of solution at bottom of the diffusion tube in gms./cc.
D = its constant density.
n = concentration of the solution at a point l cms. below the top of the tube when the steady state has been reached.
d = density of the solution at the same point.
v = velocity downwards of the water-component of the solution in the tube at the same point.
v_0 = velocity downwards of the water at the top of the the tube, where $l=0$.

Assume the area of cross-section of the tube is unity. Now 1 cc. of solution contains $(d-n)$ grammes of water, and since the same mass of water crosses each section of the tube per second in the steady state

$$v(d-n)=v_0.$$

Also, when the steady state is reached, the mass of salt passing each section of the tube per second is constant $=c$.

The change, i, in weight of the flask per second is equal to the mass of salt leaving it per second minus the mass of water entering per second, or in symbols

$$i=c-v_0.$$

Let δ represent the ratio of the change in volume produced in the solution, to the change in the amount of salt present. Then, since every second c grammes of salt escape from the flask, the volume changes by $c\delta$ cc., or $c\delta$ cc. of water enter the flask per second from the bath,

i.e., $$v_0=c\delta.$$

Thus $$v = c\delta/(d-n)$$
and $$\begin{aligned} i &= c - c\delta \\ &= c(1-\delta). \end{aligned}$$

In the previous work the mistake was made of assuming that $c\delta$ cc. of *solution* entered the flask from the tube. The error will be made quite obvious if the upper end of the tube is considered, instead of the lower; or it can be argued that if c grammes of salt leave the flask, the volume they occupied must be taken by *water*, and not by solution, for if so, some salt would return into the flask and the quantity leaving would not be c.

Under the combined actions of the movements of the liquid and diffusion it follows that

$$c = -vn + K\,dn/dl.$$

Substituting the value of v already found

$$c(d-n+n\delta)/(d-n) = K\,dn/dl \quad . \quad . \quad . \quad . \quad (1)$$

Now d is approximately a linear function of n and we make very little error if we assume that over any given range of concentration $d=1+an$, where a is some constant depending slightly on the upper limit of concentration N, and whose value may be considered to be very approximately equal to the mean slope of the true curve of d and n between the limits zero and N.

If b is written for $(1-a)$ equation (1) becomes

$$\{c+n(v_0-bc)\}/(1-bn) = K\,dn/dl,$$

the solution to which is

$$-\frac{bN}{v_0-bc} + \frac{v_0}{(v_0-bc)^2}\log_e\frac{(v_0-bc)N+c}{c} = \frac{L}{K}. \quad . \quad . \quad (2)$$

Since $(v_0-bc)N$ is small compared with c this may be written

$$\frac{N}{c}\left[1-\frac{v_0N}{2c}+\frac{v_0(v_0-bc)N^2}{3c^2}\ \ldots\right] = \frac{L}{K}. \quad . \quad . \quad (3)$$

These equations are exactly of the same form as those already published in equations (2) and (3), Proc. Phys. Soc., XXI., p. 865, 1908, where it was mentioned that the third term in the bracket in equation (3) was negligible. Owing to the correction now introduced this term is still smaller, and hence we may write

$$\frac{N}{c}\left[1-\frac{v_0N}{2c}\right] = \frac{L}{K},$$

or since $v_0=c\delta$

$$c=\frac{KN}{L}(1-\tfrac{1}{2}N\delta), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

i.e., as $c=i/(1-\delta)$

$$K=\frac{Li}{N(1-\frac{1}{2}N\delta)(1-\delta)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

It is interesting to note that this equation does not involve any knowledge of a or b or D. δ is readily obtained from the published Tables connecting density and concentration.

Thus the equations on p. 866 in the first Paper mentioned in § 1 must be corrected by omitting in every case the factor $(1-bN)$. In the second and third Papers the corresponding factor $(D-N)$ must be deleted, so that the value of K is obtained from the equation

$$K=\frac{L_1L_2}{A_1L_2-A_2L_1}\times\frac{i_1-i_2}{N(1-\frac{1}{2}N\delta)(1-\delta)}.$$

§ 4. *The Coefficient of Diffusion at Different Temperatures.*

The following table gives the results for solutions of potassium chloride and nitrate at the concentrations and temperatures mentioned. In the first column the figure 1, 2 or 3 indicates the Paper in which the result was previously published (*see* § 1), while the second column gives the number of the experiment, in order that comparison can be made and the magnitude of the correction estimated.

20 per cent. KCl or 2·7 normal or N=0·2 gm./cc.

Paper.	Experiment.	Temp. in °C.	$K\times10^5$ C.G.S.
1	24	0	1·040
1	25	0	1·054
2	4	0	1·042
2	5	13	1·391
2	6	14	1·408
3	28	18·8	1·633
3	10	18·7	1·663
3	11	18·8	1·666
2	12	23·7	1·766
2	13	23·8	1·760

10 per cent. KCl or 1·34 normal or $N=0{\cdot}1$ gm./cc.

1	26	0	0·987
1	27	0	0·989
2	7	14·6	1·412
2	8	15·8	1·479
2	9	16·9	1·507
3	8	18·6	1·545
3	9	18·7	1·545
2	14	23·8	1·785
2	15	23·9	1·782

10 per cent. KNO_3 or 1·0 normal or $N=0{\cdot}1$ gm./cc.

Paper.	Experiment.	Temp. in °C.	$K\times10^5$ C.G.S.
1	16	0	0·873
1	17	0	0·875
2	10	15·1	1·266
2	18	16·3	1·292
3	30	18·9	1·338
3	31	19·0	1·358
2	16	23·4	1·519
2	17	23·6	1·520

5 per cent. KNO_3 or 0·5 normal or $N=0{\cdot}05$ gm./cc.

1	22	0	0·885
1	23	0	0·887
2	19	16·2	1·286
2	20	16·3	1·303
3	42	18·3	1·371
3	32	19·1	1·445
3	33	19·2	1·421
2	21	25·0	1·547
2	22	25·6	1·575

A comparison of these figures with those previously published shows that the correction causes an increase in the numerical value obtained for the diffusivity which amounts to 6 or 7 per cent. in the case of 20 per cent. solutions of potassium chloride. For a similar solution of sodium chloride the correction is found to be only 4 per cent. In less concentrated solutions the correction is smaller, being roughly proportional to the concentration, and may be considered to be within the probable error of observation for all solutions weaker than 10 per cent.

§ 5. *The Temperature Coefficient.*

From the above figures the corrected values of the temperature coefficient can be calculated.

Thus if $K_t=K_0\,(1+at+bt^2)$ the values of a and b are as follows, for the four solutions mentioned :—

Salt.	Normality.	N in gm./cc.	$K_0\times10^5$	a	b
KCl	2·7	0·2	1·045	0·020	0·00050
	1·35	0·1	0·988	0·023	0·00047
KNO_3	1·0	0·1	0·874	0·023	0·00032
	0·5	0·05	0·886	0·024	0·00027

These figures indicate that in the neighbourhood of 18°C. the simpler formula approximately holds

$$K_t=K_{18}\{1+a(t-18)\},$$

where a has the value 0·024 for all these solutions.

For 2·7 normal KCl at 18°C. we find

$$K_{18}=1{\cdot}639\times10^{-5}\text{ sq. cm./sec.}$$

Dr. Griffiths by two different and independent methods obtains for a similar concentration and temperature the values 1·696 and $1{\cdot}617\times10^{-5}$ respectively. [*See* Griffiths, Proc. Phys. Soc., XXVIII., p. 255, 1916.]

§ 6. *The Coefficient of Diffusion at Different Concentrations.*

If the temperature be maintained constant during each experiment, and in the neighbourhood of 18·5°C. we obtain in the same way recalculated results for solutions of potassium chloride, potassium nitrate and sodium chloride, which are plotted graphically in Fig. 1, and, in order to save space, the values, obtained from the graphs, of the diffusivity at 18·5°C. for solutions of the concentrations indicated in the first column, are tabulated on next page.

Normality.	KCl $K \times 10^5$	KNO_3 $K \times 10^5$	NaCl. $K \times 10^5$
0·05	1·364	1·502	1·159
0·1	1·412	1·475	1·163
0·2	1·446	1·446	1·169
0·4	1·475	1·409	1·181
0·6	1·491	1·382	1·190
0·8	1·506	1·362	1·201
1·0	1·521	1·344	1·209
1·5	1·558	...	1·226
2·0	1·598	...	1·240
2·5	1·636	...	1·255
3·0	...	...	1·268

§ 7. *Diffusion at a Definite Concentration.*

In practically all the methods of determining the diffusivity of salts previous to those employed by the author and his colleagues, no attempt is made to use the steady state of diffusion. That is to say, the rate of diffusion is studied before it becomes constant, and the concentration at any point in the solution is continually changing while the experiment is in progress. As is well known, the coefficient of diffusion is not the perfectly constant quantity which Fick's law indicates; but it varies, in some cases considerably, with the concentration of the solution, so that it appears impossible in the present state of development of the theory of diffusion to say to what concentration the coefficient obtained refers.

The interpretation is less indefinite if the steady state is employed in which the concentration at each point of the solution remains quite constant while the measurements are made. These measurements give what the author has called the "mean diffusivity," K, for the solution in the diffusion tube, that is, for a solution varying in concentration from zero to a certain maximum, N, in each experiment, and from the determination of this quantity it is found possible to calculate the value of the actual diffusivity, K_n, for a solution of a definite concentration, n, in the following manner:—

The equation for c (equation (4) above) might have been obtained by making the initial assumption in equation (1) that $\delta = b$. This assumption is not strictly true, but even in the strongest solutions it only introduces a small error—probably of the order one-fifth of 1 per cent.—in the value obtained for the coefficient of diffusion, and this error is smaller the more dilute the solution.

With this approximation equation (1) becomes

$$K_n = \frac{c}{1-n\delta}\,\frac{dl}{dn}.$$

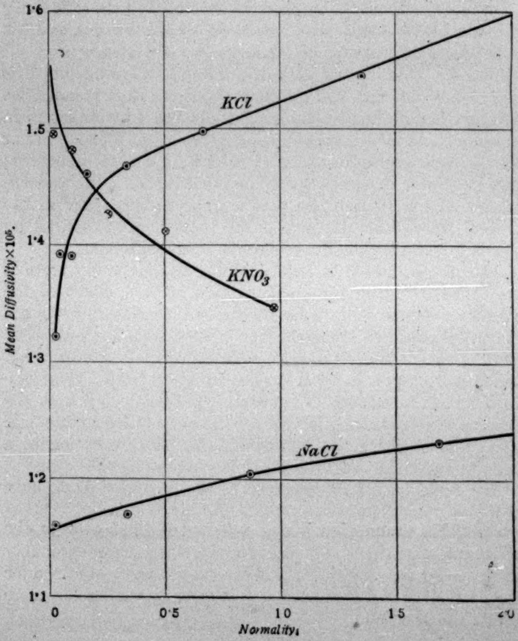
By integration this gives

$$K \times n(1-\tfrac{1}{2}n\delta) = cl,$$

where K will represent the mean diffusivity over the range of concentration 0 to n.

Differentiating this with respect to n and noting that c is a constant in any given experiment, we have

$$K + \{(1-\tfrac{1}{2}n\delta)/(1-n\delta)\}n\,.\,dK/dn = \frac{c}{1-n\delta}\,\frac{dl}{dn} = K_n. \quad . \quad (6)$$



So that we can find the value of the coefficient, K_n, at any definite concentration, n, from the above expression involving a measurement of the slope of the experimental curves shown in Fig. 1.

The values of the coefficient of diffusion at 18·5°C. in C.G.S. units (sq. cm./sec.) thus obtained at the definite concentrations indicated in the first column are tabulated below :—

Normality.	KCl. $K_n \times 10^5$	KNO_3 $K_n \times 10^5$	NaCl. $K_n \times 10^5$
0·05	1·442	1·464	1·164
0·1	1·459	1·435	1·170
0·2	1·488	1·402	1·181
0·4	1·511	1·349	1·202
0·6	1·534	1·315	1·220
0·8	1·564	1·285	1·238
1·0	1·598	1·259	1·248
1·5	1·675	...	1·273
2·0	1·755	...	1·397
2·5	1·834	...	1·327
3·0	...	...	1·355

In a previous Paper (Proc. Phys. Soc., XXVII., p. 64, 1914) the author attempted to obtain this quantity K_n but was obliged to make certain assumptions as to the movement of the liquid which were not strictly in accordance with the facts, owing to mathematical difficulties in the treatment of the real conditions obtaining in the experimental work, but the simpler formulæ now found to be correct enable the actual diffusivity at any definite concentration to be evaluated for the first time without making the assumptions just referred to. As has already been mentioned, this calculation appears to be impossible unless the steady state is employed, and represents a new departure in work on diffusion.

ABSTRACT.

The Paper contains the results of the experiments described by the author in Proc. Phys. Soc. Lond., XXI., p. 374, 1908; XXIV., p. 40, 1911; and XXVII., p. 56, 1914, collected and recalculated in accordance with a theoretical correction recently communicated by Dr. Griffiths (Proc. Phys. Soc., XXVIII., p. 255, 1916).

It is found that in the solutions employed the correction is not considerable, except in the case of the strongest solutions of KCl (2·7 normal), where it amounts to 6 per cent.

The Paper contains the corrected theory of the method, and the value of the coefficient of diffusion is tabulated at different limiting

concentrations and at various temperatures; it concludes with a new method of calculating the coefficient at a definite concentration from the experimental results, which appears to be more accurate and free from questionable assumptions than previous calculations of a similar nature.

DISCUSSION.

Prof. G. W. O. **Howe** asked why the concentration was constant at the bottom of the diffusion tube in the author's apparatus.

Prof. **Boys** thought this was an important point. He presumed that the upper bulb was sufficiently large that for the purposes of the experiment it could be regarded as an infinite reservoir. A steady state would then be maintained by the circulation through the side tubes, dilute solution rising up one into the reservoir, while more concentrated solution passed down the other.

Dr. R. S. **Willows** asked why the coefficient of diffusion was greater for strong than for weak solutions in the case of some salts, while the reverse was true for other salts.

Mr. **Clack**, in reply to Dr. Willows, said he did not know of any reason, but Dr. S. W. J. Smith had told him that that was what would be expected from conductivity data.

IV.—*On the Measurement of the Thomson Effect in Wires.* By H. Redmayne Nettleton, *M.Sc., Assistant Lecturer in Physics at Birkbeck College.*

Received October 25, 1916.

I. Introduction.

The object of this research is to develop a method of measuring the Thomson Effect in wires for which also the thermoelectric powers may at the same time be found.

It is well known that the application of thermodynamics to thermocouples leads to the two standard equations :—

$$\Pi = T \cdot \frac{dE}{dT}, \quad \ldots \ldots \ldots \quad (1)$$

where Π is the Peltier coefficient at the absolute temperature T and dE/dT the thermo-electric power ; and

$$\sigma_a - \sigma_b = -T \cdot \frac{d^2E}{dT^2}, \quad \ldots \ldots \quad (2)$$

where σ_a and σ_b are the respective Thomson effects in the metals comprising the thermocouple.

Equation (1) was first investigated experimentally by Campbell[1] and a little later verified by Jahn[2] with the ice-calorimeter, and since the further development of the measurement of the Peltier effect by Lecher,[3] Cermack,[4] Barker,[5] Oosterhuis,[6] and Callendar[7] it may be said that this equation has been substantiated beyond all doubt. Szarvassi,[8] indeed, regards this equation as the fundamental formula of thermo-electric phenomena and Callendar uses it for his values of Π in measuring heat by the radio-balance.

Equation (2), however, has never been verified. On the contrary, the best measurements of the Thomson effect seem to throw great doubt on the equation (Szarvassi), and so grave is this divergence in the case of the iron-mercury thermocouple

[1] Campbell, "Proc. Roy. Soc. Edin.", p. 387, 1887.

[2] H. Jahn, "Wied. Ann." 34, 755, 1888.

[3] E. Lecher, "Wien. Ber." II.A., 115, p. 1505, 1906.

[4] P. Cermack, "Wien. Ber." II.A, 116, pp. 657 and 1135, 1907.

[5] Barker, "Phys. Rev." XXXI. (4), 321, 1910.

[6] Oosterhuis, "Arch. Néer" Ser III A II, 1912.

[7] Callendar, "Proc. Phys. Soc. Lond." 23. I., Dec., 1910.

[8] Szarvassi, "Ann. der Phys." 17, 248, 1905.

treated of by Schoute,[1] Aalderink[2] and Oosterhuis[3] that the last-named experimenter raises the whole question of the energy of a thermo-electric circuit and states that the Peltier and Thomson effects together do not nearly suffice to supply the energy of the thermo-electric current.

Before, however, any general conclusions can be made on this matter a vast amount of experimental research on the Thomson effect is needed, for it must be remembered that not only is the measurement of the Thomson coefficient one of considerable difficulty, and that impurities have a very great effect, but that many of the methods employed—especially the older ones—are open to severe criticism, particularly when viewed in the light of Verdet's[4] differential equation. As an illustration of what equation (2) leads to we have the following: The thermocouple iron-german silver between 0°C.–200°C. gives a curve for thermo-electric force and temperature which is remarkably linear (Campbell[5], Stewart[6]); iron-nickel and iron-constantan behave somewhat similarly. Hence over this temperature range d^2E/dT^2 is nearly zero and the Thomson effects in iron, german silver, nickel and constantan should approach equality. Now, in a very careful research, Aalderink[7] has shown that the value of σ in iron changes sign at 95°C. and is given by

$$\sigma=(95-t)\times 4{\cdot}9\times 10^{-8}\text{ cal. per coulomb.}$$

Do, then, these other metals show any similar remarkable change of sign or equality of value?

It was thus the author's first intention to measure the Thomson effect in wires of the high-resistance alloys and at the same time measure their thermo-electric powers when combined in pairs. As, however, the Thomson effect is not known with any certainty even in metals like bismuth and platinum in which it is largest, and as the method here described has proved rapid and sensitive, it is now proposed to measure the effect in as many metals as possible. In the present communication it is proposed to deal only with the method itself, investigating the errors, sensibility, and best conditions of experiment.

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[1] Schoute, "Arch. Néer." (2), 12, 175, 1907.
[2] Aalderink, "Arch. Néer." (2), 15, 321, 1901.
[3] Oosterhuis, *loc. cit.*
[4] Verdet, "Theorie Méchanique de la Chaleur."
[5] Campbell, "Proc. Roy. Soc. Edin.," 1882.
[6] Stewart, "Phil. Trans." A., p. 569, 1893.
[7] Aalderink, *loc. cit.*

2. The Proposed Method and Its Relation to Other Methods.

The many methods which have been employed to measure the Thomson effect may be classified under four headings :—

(*a*) Battelli's method[1] in which σ is measured directly by the rate of rise of temperature of a calorimeter of known total water equivalent.

(*b*) Haga's method,[2] of which there are many modifications, in which the effect of Thomson heat is compared with the effect of a known amount of Joulean heat.

(*c*) Callendar's method,[3] carried out by King, in which σ is deduced from the variation of resistance with temperature of the wire under test.

(*d*) The impressed velocity method of the author,[4] in which the Thomson effect is neutralised or compared with the effect of a slow uniform movement of the conductor.

Battelli's method is chiefly of historic interest, though there is no doubt it might be modernised ; not only were his experiments complicated by working during the variable state ; but a study of the distribution of temperature down the bar leads to the conclusion that much of the Thomson heat produced outside the calorimeter contributes by conductivity to its rise of temperature. This method which is criticised by Callendar[5] and King for other reasons also lacks sensibility.

Method (*d*) is suitable only for fluids.

Method (*c*) alone has been applied to a wire ; it is of great interest though complicated by the insertion of potential leads. By its nature it is unsuited to the alloys of small temperature coefficient.

Method (*b*), Haga's method, is by far the most generally employed, and undoubtedly of the greatest importance, modifications of it being used by such recent experimenters as Laws,[6] Lecher,[7] Schoute,[8] Berg,[9] and Aalderink.[10] When employing this method it must always be remembered that

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[1] Battelli, "Accad. delle Sci. di. Torino," Atti XXII., p. 548, 1887.

[2] Haga, "Ann. de l'ecole Polyt de Delft," I., p. 145, 1885; III., p. 43, 1886.

[3] King, "Amer. Acad. Proc." XXXIII., p. 353, 1898.

[4] Nettleton, "Proc. Phys. Soc. Lond.," Dec., 1912.

[5] Callendar, "Encyclo. Brit." Thermo-electricity.

[6] Laws, "Phil. Mag." VII., p. 560, 1904.

[7] Lecher, "Ann. de Physik." XIX., p. 853, 1906.

[8] Schoute, *loc. cit.*

[9] Berg, "Ann. de Physik." XXXII.-XXXIII., p. 477, 1910.

[10] Aalderink, *loc. cit.*

Thomson heat and increments of Joulean heat are only propagated similarly when small and of the same order of magnitude, and that close attention must be paid to the emissivity conditions.

In perhaps its most modern general form Haga's method is produced as follows: A current of C amperes is passed down a bar in which a temperature gradient is maintained. This current is reversed and the ultimate change of temperature $\Delta\theta_T$ is recorded as shown by a thermocouple in the middle of the gradient. The current is then raised or diminished by an amount c amperes, and the difference of temperature $\Delta\theta_J$ is observed between the average temperature given by $\pm C$ and the average temperature given by $\pm(C+c)$. It can then be shown that, if J is the number of joules equal to one calorie, and r the resistance per unit length of the bar, we have, under suitable conditions, with sufficient accuracy:

$$\frac{2C\sigma\,.\,d\theta/dx}{c^2r/J}=\frac{\Delta\theta_T}{\Delta\theta_J}.$$

The temperature gradient $d\theta/dx$ is usually found by a thermocouple which may be inserted in two or more positions along the bar; usually two bars side by side are used to advantage instead of one.

When we attempt to apply Haga's method to a wire we are confronted with additional difficulties, for it becomes less easy to measure both the temperature variation at the middle and the temperature gradient itself without altering the resistance per unit length of the wire and disturbing the linear heat flow. Briefly, these difficulties are overcome thus:—

(1) Instead of measuring the temperature gradient in any neighbourhood, the difference of temperature between the "ends" of the wire—the same ends across which the electrical resistance is determined—is measured in a similar way to that used by Gray[1] in his heat conductivity experiments on wires. Thus the ratio R/U of the equation below is obtained.

(2) The temperature at the middle is not allowed to alter, Haga's method being employed in the null form used by Berg[2] in his work on plates, and the detector of temperature is a number of turns of the thinnest insulated copper wire used as a resistance thermometer. This method, which Starling[3] first

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[1] Gray, "Phil. Trans." A. Part I., p. 165, 1895.

[2] Berg, *loc. cit.*

[3] Starling, "Nature," Feb. 16, 1911.

used to demonstrate the Thomson effect in a red hot iron bar conveying 10 amperes, is found to be sufficiently sensitive with No. 44 wire for measuring absolutely the Thomson effect in wires conveying 3 amperes and heated by steam. The diameters of the bare and double silk-covered detecting wire are only about 0·07 mm. and 0·14 mm. respectively, and the thread-like coil may be wound uniformly on the lathe over the thicker experimental wire (gauge 18, diameter 1·22 mm.), under investigation.

The apparatus used is very simple and will readily be understood from Fig. 1. The wire *W* under test passes right through the constant temperature sources *H* and *C*. *H* is a cylindrical copper vessel 8 cm. long and of 5 cm. diameter, fitted with vapour inlet and outlet tubes and closed at *D* by a rubber bung through which the experimental wire *W* passes easily before being cemented up. *C* is a rectangular copper vessel 5 cm. square by 10 cm. long, which will hold water or ice. The sides *A*, *B*, of *H* and *C* respectively are 5 mm. thick and drilled with holes of 1 mm. diameter to admit thermo-junctions at *a* and *b* of iron and constantan wire which record the difference of temperature *U* between the "ends" of the wire under test. The holes *a*, *b*, also serve to admit potential leads for measuring the resistance of the wire between its "ends."

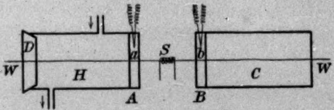

Fig. 1.

Great care is taken in passing and fixing the wire through *A* and *B*, as no solder must appear internally, and yet good contact is essential. The side *A* is made as shown in Fig. 2, the narrow portion being drilled at *N* with a hole through which the wire just passes. Solder is applied externally through *K* with a special iron, care being taken that it adheres to the copper as well as the wire under test. *B* is similarly made, the solder being applied within the vessel *C*. With practice this fixing of the wire may be done very neatly. The internal length between *A* and *B* was usually between 4 cm. and 5 cm., and over about one-third of this length the detecting coil *S* of copper wire, gauge 44, already alluded to, was symmetrically wound (previous to fixing the wire), the centre of symmetry lying midway between A and B. The ends of the coil S were brought to terminals on the base board from which they were connected by thicker copper wire, gauge 18, to a resistance bridge arranged as described in section (4) b of this Paper. The resistance of S under the conditions of experiment varied from 5 ohms to 8 ohms. Cotton wool was carefully wound round the operative part of the wire as well as round the temperature sources, H and C. It serves to allay convection currents, and for this reason is essential and, as indicated by the resistance coil S, very considerably diminishes the emissivity loss.

Fundamentally, an experiment is performed thus: Water, either ice-cold or at ordinary temperature, being in the vessel

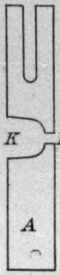

FIG. 2.

C and steam being circulated through H, a steady current C_1 of from 2·5 to 3·5 amperes is passed through the wire in either direction. If this current is merely reversed the coil S shows a change of resistance, which measures the sensibility, due to a change in the rate of production of Thomson heat; but if, at the same moment that the reversal is made, the current is slightly altered to an appropriate value, C_2, a complete balance may be obtained between the alterations of Joulean and Thomson heat, the coil S then showing no change of temperature. As shown in section 3 below, we then have, under suitable conditions, with sufficient accuracy :—

$$\sigma=\frac{R}{JU}(C_2-C_1),$$

where U is the difference of temperature between the ends of the wire, R the actual resistance of the wire between these ends under the temperature conditions of the experiment, and J the number of joules equal to one calorie. The Thomson effect σ will then be expressed in calories per coulomb per degree centigrade and, as shown below, its value will be that at the mean temperature $U/2$. The difference of the up-and-down currents, C_2-C_1, called the balancing current, can be measured directly with a milliammeter, or on a Cambridge thermoelectric potentiometer, this measurement and that of R being described in section 4 of this Paper. Owing to the shortness of the wire and the fact that for a balance the temperature at the middle remains constant, such a balance may be found quickly—new steady states being obtained in a few minutes.

In contrasting the conditions of these experiments with those for Gray's conductivity experiments we observe in the present research :—

1. That the quantity of heat passing down the wire has not to be measured, that a small emissivity loss is permitted by the theory, and that error due to the temperature gradient not being quite linear is thus of smaller magnitude. Moreover, the temperature gradient in the strict sense is not required at the middle of the wire, but rather the term U/L of the next section. Probably, therefore, in the present method, a longer wire in proportion to its diameter may be used.

2. Greater care must be taken in this method than in Gray's method in passing the wire through the constant temperature sources. The presence of any solder upon the operative part of the wire would not only alter the resistance per unit length, but might complicate also by the production of Peltier heat. The production of Peltier heat at the actual contact of the wire with the thick copper pieces A and B, presenting such contrast in cross-section with the wire, can easily be shown to have no sensible effect; and the relation between the thermal capacities and thermal conductivities of the thick portions A and B, and those of the wire, together with the manner of passing the wire through and securing its contact, fully justify A and B being regarded as constant temperature ends. With every wire a practical test was made to show that no sensible change of resistance of the detector S resulted on reversing a steady current when H and C were maintained at the same temperature.

3. The Theory of the Method.

(*a*) *Preliminary Treatment. Ideal Conditions.*

Considering a rod or wire under a temperature gradient, arranged as in Fig. 1, above let all temperatures and temperature-coefficients be dated from the fixed " cold end " taken as zero, and let U be the temperature of the hot end a distance L away.

Let C=electric current in amperes traversing the wire from hot to cold.
S=specific resistance of the material.
K=thermal conductivity of the material.
A=area of cross-section of wire.
p=perimeter of wire.
α=temperature coefficient of resistance.
σ=Thomson effect, assumed positive and at first constant.
E=heat loss per square centimetre per sec. per 1°C. excess temperature above the enclosure.
θ_0=temperature of enclosure.
J=number of joules equivalent to one calorie.
θ=temperature at a distance x from the hot end.

The differential equation expressing the distribution of temperature may be written :—

$$\frac{d^2\theta}{dx^2}+a\cdot\frac{d\theta}{dx}+b\theta=c,$$

where the Thomson heat term $a=-C\sigma/KA$

the emissivity term $b=\left[\frac{C^2S\alpha}{JA}-Ep\right]/KA,$

the Joulean heat term $c=-\left[\frac{C^2S}{JA}+Ep\theta_0\right]/KA.$

The general solution under the present end conditions is :—

$$\theta=\frac{c}{b\sinh\lambda L}\left[e^{\frac{-ax}{2}}\sinh\lambda(x-L)-e^{\frac{-a}{2}(x-L)}\sinh\lambda x\right]+\frac{c}{b}$$
$$-\frac{U}{\sinh\lambda L}e^{\frac{-ax}{2}}\sinh\lambda(x-L),$$

where $\lambda=\sqrt{(a^2-4b)}/2.$

But if, as is nearly always justifiable, $aL/2$ may be taken as small compared with unity, and if in addition the emissivity term b is so small that terms in $a\,.\,b$ and $c\,.\,b$ may be neglected in comparison with a and c respectively, we have

$$\theta=U\left(\frac{L-x}{L}\right)\left(1-\frac{ax}{2}\right)+U\left(\frac{L-x}{L}\right)\frac{bx}{6}(2L-x)-\tfrac{1}{2}cx(L-x).$$

When a current C_1 is passing from hot to cold put $\theta=\theta_1$; $c=c_1$; $b=b_1$; and $a=a_1=-C_1\sigma/KA$; and when a current C_2 is passing from cold to hot put $\theta=\theta_2$; $c=c_2$; $b=b_2$; and $a=a_2=+C_2\sigma/KA$.

The temperature change on reversing the current C_1 and increasing its value to C_2 is thus given by:—

$$d\theta=U\left(\frac{L-x}{L}\right)\frac{x}{2}(-a_2+a_1)+U\left(\frac{L-x}{L}\right)\frac{x}{6}(2L-x)(b_2-b_1)$$
$$-\tfrac{1}{2}x(L-x)(c_2-c_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If this is zero at the middle where $x=L/2$, we obtain:

$$\sigma=\frac{LS}{UJA}(C_2-C_1)(1+\tfrac{1}{2}aU) \quad . \quad . \quad . \quad . \quad (2\text{A})$$

$$=\frac{R_0}{JU}(C_2-C_1)\left(1+a\frac{U}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad (2\text{B})$$

$$=\frac{R}{JU}(C_2-C_1) \text{ very approximately,} \quad . \quad (2\text{C})$$

where R_0 is the resistance of the actual wire when at the temperature of the cold end and R its resistance when under temperature gradient.

Conversely when the current C_2 is so chosen that at the middle $d\theta$=zero and relation (2) holds, we have (on substituting in equation (1)) for $d\theta$ elsewhere:—

$$d\theta=U\left(\frac{L-x}{L}\right)\frac{x}{2}\left(\frac{L-2x}{6}\right)(b_2-b_1)$$

$$=U\left(\frac{L-x}{L}\right)\frac{x}{12}(L-2x)\frac{Sa}{JA\,.\,KA}({C^2}_2-{C^2}_1).$$

Thus, if a is zero, the balance is not only in the middle but everywhere. If a is not zero, the deviation is symmetrical about the middle and can easily be shown to be very slight, compared with the temperature changes which result on merely reversing the current, but not increasing its magnitude. Thus under the conditions assumed above the detecting coil may have appreciable length if symmetrically wound about the middle; moreover, errors due to lack of perfect symmetry will be very small.

(*b*) *Complete Theory: Special Conditions Necessary for Wires.*

While in general, in work on the Thomson effect, the "a" term, equal to $-C\sigma/KA$, may be taken as small compared with unity, on the other hand, the neglect of products of the "a," "b" and "c" terms is often not justifiable. When dealing with wires it is more necessary than ever to pay special attention to these points, and the following form the chief consideration of this section :—

(i.) The effect of a temperature coefficient of the Thomson effect.

(ii.) The effect of a rise of the emissivity loss on both the accuracy and sensibility.

(iii.) The effect of a rise of Joulean heat.

(iv.) The effect of the length of the detecting coil when product terms are considered and σ is a variable.

(v.) The effect of a rise of the Thomson heat or "a" term towards unity.

Assuming that over the range of temperature between the hot and cold ends the variation of σ is linear, we may write, dating temperatures from the cold end as before

$$\sigma_\theta=\sigma(1+\beta\theta).$$

and the differential equation of temperature distribution becomes—

$$\frac{d^2\theta}{dx^2}+a(1+\beta\theta)\frac{d\theta}{dx}+b\theta=c.$$

Still assuming a to be small compared with unity, but otherwise carrying the expansion much further, we obtain the solution—

$$\begin{aligned}\theta=U\left(\frac{L-x}{L}\right)\left(1-\frac{ax}{2}\right)-\frac{U^2a\beta}{6L^2}x(L-x)(2L-x)+\frac{Ub}{6L}x(L-x)(2L-x)\\ -\frac{Uab}{12L}x^2(L-x)(2L-x)-\tfrac{1}{2}cx(L-x)+\frac{bc}{24}x(L-x)(x^2-xL-L^2)\\ -\frac{ac}{12}x(L-x)(L-2x)+\frac{abc}{720}x(L-x)(L-2x)(6x^2-6Lx-7L^2) \quad (3)\end{aligned}$$

and hence, using the same notation as before, on changing from a current $+C_1$ to a current $-C_2$, we obtain :—

$$\begin{aligned}d\theta=\frac{U}{2L}x(L-x)(-a_2+a_1)+\frac{U^2\beta}{6L^2}x(L-x)(2L-x)(-a_2+a_1)\\ +\frac{Ux}{6L}(L-x)(2L-x)(b_2-b_1)+\frac{U}{12L}x^2(L-x)(2L-x)(-a_2b_2+a_1b_1)\\ +\tfrac{1}{2}x(L-x)(-c_2+c_1)+\frac{x}{24}(L-x)(x^2-xL-L^2)(b_2c_2-b_1c_1)\\ +\frac{x}{12}(L-x)(L-2x)(-a_2c_2+a_1c_1)+\frac{x}{720}(L-x)(L-2x)\\ (6x^2-6Lx-7L^2)(a_2b_2c_2-a_1b_1c_1) \quad . \quad (4)\end{aligned}$$

Equating this to zero and simplifying :—

$$(a_2-a_1)\left\{1+\frac{\beta U(L-x)}{L}-\frac{U\beta}{3L}(L-2x)\right\}$$

$$=\frac{2L-x}{3}(b_2-b_1)+\frac{L}{U}(c_1-c_2)+\frac{x}{6}(2L-x)(a_1b_1-a_2b_2)$$

$$+\frac{L}{12\,.\,U}(x^2-xL-L^2)(b_2c_2-b_1c_1)+\frac{L}{6U}(L-2x)(a_1c_1-a_2c_2)$$

$$+\frac{L}{360\,.\,U}(L-2x)(6x^2-6Lx-7L^2)(a_2b_2c_2-a_1b_1c_1)\quad .\quad .\quad (5)$$

An examination of the left-hand side of this equation at once shows that a thermo-junction at the middle of the gradient or a detecting coil symmetrically wound about the middle will respond to the value of σ at this middle temperature. Putting $x=L/2$ we find :

$$(a_2-a_1)(1+\beta\theta)=\frac{L}{2}(b_2-b_1)-\frac{L}{U}(c_2-c_1)+\frac{L^2}{8}(a_1b_1-a_2b_2)$$

$$+\frac{5L^3}{48\,.\,U}(b_1c_1-b_2c_2)\ .\quad (5\text{A})$$

which, on substituting for the a, b and c terms and writing b for $(b_1+b_2)/2$ and $c=(c_1+c_2)/2$ becomes very approximately :—

$$\sigma_\theta=\frac{R_0}{JU}(C_2-C_1)\left(1+\frac{aU}{2}\right)-\frac{b\sigma L^2}{8}+\frac{5L^2}{48}\cdot\frac{R_0}{JU}(C_2-C_1)(b-ac)$$

or very approximately.

$$\sigma_\theta=\frac{R_0}{JU}(C\ -C_1)\left(1+\frac{aU}{2}\right)-\frac{b\sigma L^2}{48}\quad .\quad .\quad .\quad .\quad .\quad (6)$$

Hence, as b is negative, if the length of wire is increased or the protection diminished, so that b becomes of appreciable value, the values of σ obtained by neglecting it will become too low. The effect of the guard-ring of cotton wool in diminishing b is very great—as may be judged by the change of resistance of the detecting coil on removing it—but experiments were performed with constantan wire of different lengths (the metal for which b is greatest) to show that in the main experiments the lengths of wire were sufficiently short for the correcting term to be neglected.

The effect of a rise of emissivity is also to diminish the sensibility which is measured by the change of temperature at the middle, on reversing, but not altering, the magnitude of a mean current $C=(C_1+C_2)/2$. Putting $a_2=-a_1=+C\sigma/KA$,

$b_1=b_2=b$, $c_1=c_2=c$, and $x=L/2$ in equation (4), and neglecting variations of σ with temperature, we obtain for the sensibility :—

$$d\theta=\frac{ULC\sigma}{4KA}\left(1+\frac{bL^2}{8}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Though an increase in L within the limits allowed by the emissivity loss increases the sensibility any such benefit is in great part lost by the increase in time needed for the attainment of the steady state. An increase in U is also undesirable and unnecessary, the smallness of A giving satisfactory sensibility even with moderate currents.

From equation (5A) it is easy to see that a rise of the c term embracing Joulean heat is—unlike a rise in the b term—quite unimportant both from the standpoints of error and sensibility.

It is important to study the effect of increasing the length of the detecting resistance coil beyond the immediate neighbourhood of the middle of the wire, thereby obtaining a greater absolute change of resistance on reversing an electric current through the wire under test.

Let d=length of detecting coil symmetrically wound about the middle. Then the average temperature to which it corresponds is given by :—

$$\varphi=\frac{1}{d}\int_{\frac{L-d}{2}}^{\frac{L+d}{2}} \theta \, . \, dx.$$

Passing a current C_1 in the positive direction, writing $a=a_1$, $b=b_1$, $c=c_1$, and integrating the expansion for θ of equation (3) we have :—

$$\begin{aligned}\varphi_1=\frac{U}{2}-a_1\frac{UL}{8}\left(1-\frac{d^2}{3L^2}\right)+b_1\,.\,\frac{UL^2}{16}\left(1-\frac{d^2}{3L^2}\right)\\ -\tfrac{1}{8}c_1L^2\left(1-\frac{d^2}{3L^2}\right)-\frac{a_1b_1U}{12L}\left(\frac{45L^4-20L^2d^2+3d^4}{240}\right)\\ -b_1c_1\left(\frac{25L^4-10L^2d^2+d^4}{1920}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)\end{aligned}$$

the term a_1 including β the Thomson effect coefficient and standing for $\frac{-C_1\sigma}{KA}\left(1+\beta\,.\,\frac{U}{2}\right)$. A similar expression is obtained for φ_2 on passing a current C_2 in the opposite direction, whence

$$\varphi_2-\varphi_1=-(a_2-a_1)\frac{UL}{8}\left(1-\frac{d^2}{3L^2}\right)+(b_2-b_1)\frac{UL^2}{16}\left(1-\frac{d^2}{3L^2}\right)$$
$$-(c_2-c_1)\frac{L^2}{8}\left(1-\frac{d^2}{3L^2}\right)-(a_2b_2-a_1b_1)\frac{U}{12L}\left(\frac{45L^4-20L^2d^2+3d^4}{240}\right)$$
$$-(b_2c_2-b_1c_1)\left(\frac{25L^4-10L^2d^2+d^4}{1920}\right). \quad . \quad . \quad . \quad (9)$$

Equating to zero and dividing throughout by $\frac{UL}{8}\left(1-\frac{d^2}{3L^2}\right)$ we have :—

$$a_2-a_1=\frac{L}{2}(b_2-b_1)-\frac{L}{U}(c_2-c_1)-\frac{1}{12\,.\,L^2}\cdot\frac{1}{30}(a_2b_2-a_1b_1)\left(\frac{45L^4-20L^2d^2+3d^4}{1-d^2/3L^2}\right)$$
$$-\frac{1}{LU}\cdot\frac{1}{240}(b_2c_2-b_1c_1)\left(\frac{25L^4-10L^2d^2+d^4}{1-d^2/3L^2}\right).$$

Substituting for the a, b and c terms, and in the last two terms neglecting α and β and writing b for $(b_1+b_2)/2$ we have the very approximate expression :—

$$\sigma_\theta=\frac{R_0}{JU}(C_2-C_1)\left(1+\frac{aU}{2}\right)-\frac{b\sigma}{720}\frac{1}{L^2}\left(\frac{15L^4-10L^2d^2+3d^4}{1-d^2/3L^2}\right). \quad (10)$$

The correcting term in b becomes $-b\sigma L^2/48$ for $d=$zero, $-b\sigma L^2/50$ for $d=L/3$, $-b\sigma L^2/52$ for $d=L/2$, and $-b\sigma L^2/60$ for $d=L$. The effect of the detecting coil is thus small but helpful in diminishing the effect of emissivity.

The assumption that the a term, *i.e.*, $C\sigma/KA$ is small compared with unity is, of course, justifiable in nearly all ordinary work where rods are used. Though L is small in the case of wires, the smallness of A is such as to render an investigation most desirable. The general case is complicated, but, by neglecting β the Thomson effect temperature coefficient, and assuming b the emissivity loss to be rendered negligible, the effect of a rise of $La/2$ may be studied. The general solution for θ is then :—

$$\theta=U\left(\frac{e^{-ax}-e^{-aL}}{1-e^{-aL}}\right)+\frac{cx}{a}-\frac{cL}{a}\left(\frac{1-e^{-ax}}{1-e^{-aL}}\right).$$

The author is indebted to Miss M. L. Humby, B.Sc., for the following solution obtained by the use of Bernouilli's numbers :

$$\theta=\frac{U(L-x)}{L}\left\{1-\frac{x}{2}a-\frac{x(L-2x)}{12}a^2+\frac{x^2(L-x)}{24}a^3\right.$$
$$\left.+\frac{x(L-2x)(L^2+3Lx-3x^2)}{720}a^4+\ldots\ldots\right\}$$
$$+c(L-x)\left\{-\frac{x}{2}-\frac{x(L-2x)}{12}\,.\,a+\frac{x^2(L-x)}{24}\,.\,a^2\right.$$
$$\left.+\frac{x(L-2x)(L^2+3Lx-3x^2)}{720}\,.\,a^3+\ldots\ldots\right\}$$

With the usual notation we may obtain $\theta_2-\theta$, the difference of temperature which results on changing from a current $+C_1$ to a current $-C_2$, and equating this to zero and simplifying we find :—

$$a_2-a_1=-\frac{L-2x}{6}(a_2-a_1)(a_2+a_1)+\frac{x(L-x)}{12}(a_2-a_1)$$
$$(a_2^2+a_1a_2+a_1^2)$$
$$+\frac{(L-2x)(L^2+3Lx-3x^2)}{360}(a_2^2+a_1^2)(a_2-a_1)(a_2+a_1)$$
$$-\frac{L}{U}(c_2-c_1)$$
$$-\frac{L}{U}\left(\frac{L-2x}{6}\right)(a_2c_2-a_1c_1)+\frac{L}{U}x\left(\frac{L-x}{12}\right)(a_2^2c_2-a_1^2c_1)$$
$$+\frac{L}{U}\,\frac{(L-2x)(L^2+3Lx-3x^2)}{360}(a_2^3c_2-a_1^3c_1).$$

Putting $x=L/2$ and causing symmetrical terms about the middle to vanish, we find on substituting for a and c :—

$$\sigma=\frac{R}{JU}(C_2-C_1)+\frac{L^2}{48}\cdot\frac{\sigma^3}{K^2A^2}(C_2^2-C_1C_2+C_1^2)-\frac{L^2}{48}\cdot\frac{R}{JU}\cdot\frac{\sigma^2}{K^2A^2}$$
$$(C_2-C_1)(C_2^2+C_1^2)$$

which leads to :—

$$\sigma=\frac{R}{JU}(C_2-C_1)-\frac{L^2}{48}\sigma^3\frac{C_1C_2}{K^2A^2}.$$

Calling $C_1C_2/K^2A^2=C^2/K^2A^2$, where $C=\sqrt{C_1C_2}$, and $a=C\sigma/KA$, we obtain :—

$$\sigma\left(1+a^2\,.\,\frac{L^2}{48}\right)=\frac{R}{JU}(C_2-C_1).\quad\ldots\ldots\quad(11)$$

So an error due to the rise of a or $C\sigma/KA$ towards unity is not important until $L^2a^2/48$ rises towards unity.

4. THE PRACTICE OF THE METHOD.

(*a*) *The Measurement of C_2-C_1 the Balancing Current.*

The method of measuring the excess current C_2-C_1 which will just balance the change of Thomson heat on reversing the main current will readily be understood from Fig. 3 below.

The main circuit consists of accumulators (10 or 12 volts), an ammeter, rheostats, the reversing switch K_1, the apparatus E with the wire under test and the special resistance R. The reversing switch K_1 was specially designed to be of negligible resistance so that the current should not alter appreciably on making a reversal even under low voltages—most mercury commutators being untrustworthy in this respect. The special resistance frame R consists of two thick constantan wires, each of about 1 ohm resistance stretched across stout copper pieces well supplied with terminals as shown. Its object is to supply the main circuit with two exactly equal resistances, the one including the experimental wire and extending from A to B in the diagram, and the other from B to C. By means of the special switch K_2 of negligible resistance either of these equal resistances can be shunted through the branch circuit M. This latter consists of a resistance box, a rheostat, a milliammeter, and a standard 0·2 ohm, with potential leads for calibrating the milliammeter directly on a Cambridge thermoelectric potentiometer reading to 30 millivolts: on the same

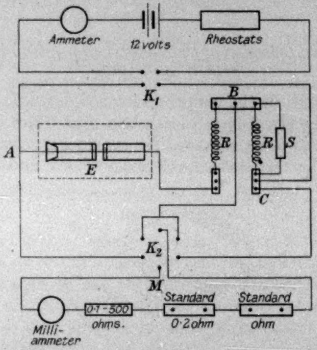

FIG. 3.

potentiometer, the 0·2 ohm could be compared with a standard 1 ohm. The equality of the two resistances A–B and B-C can be tested by the equality of full deflections of the milliammeter for both sides of the rocking key K_2—the fine adjustment being effected with the aid of the high resistance shunt rheostat S. As either A–B or B-C is always shunted, the current through the main circuit will remain constant, no matter in which direction it is flowing, and on transferring the shunt circuit M from AB to BC by the rocker K_2 the current flowing through E the wire under test will be greater than before by the reading of the milliammeter.

The keys K_1 and K_2 were side by side and could be switched over simultaneously, in which case in general the detecting coil S of Fig. 1 would indicate a change of temperature unless the changes of Joulean and Thomson heat were perfectly balanced. The milliamperes required for a balance give $C_2 - C_1$ of the formula for σ.

(b) *The Detection of Change of Temperature in the Middle of the Wire.*

The terminals of the copper resistance detector S of Fig. 1 were joined by thick copper leads to a Post Office box with 10 and 1,000 ratio arms arranged for maximum sensibility as in Fig. 4. The battery was a 2-volt accumulator, and was allowed to give a continuous current throughout the experiment so that the slight heating effect in the thermometer coil S should be constant. By balancing first to the nearest ohm and then shunting the 10 ratio arm with a high resistance it could easily be arranged that on depressing the galvanometer key no deflection resulted. On reversing the main current through the wire under test and switching over K_2 of Fig. 3 simultaneously, and again, after a minute, depressing the galvanometer

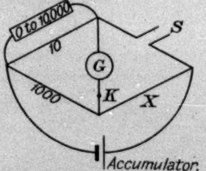

FIG. 4.

key, there was in general a deflection unless the Thomson heat was exactly balanced by the small current registered by the milliammeter. The galvanometer was a low resistance quartz fibre instrument of the Broca type, 1 micro-ampere, producing some 13 cm. deflection on the distant scale.

(c) *The Measurement of U the Temperature Difference Between the Ends of the Wire.*

The temperature difference U will be somewhat smaller than the difference of temperature between the steam in H and the water in C of Fig. 1. It was measured by an iron-constantan thermocouple insulated with special varnish, the junctions being well inserted in the holes a and b of Fig. 1, which contain a little mercury. The thermocouple gave 54·38 microvolts per degree between 0°C. and 100·15°C. and 54·92 microvolts per degree between 21·64°C. and 100·15°C. The constantan wires were joined together and the iron wires to copper leads, these latter junctions lying side by side in tubes containing paraffin oil and immersed in a large water bath. The copper leads were joined to the Cambridge thermo-electric potentiometer and the microvolts read off directly. U only differed by 0·2°C. to 0·3°C. from the temperature interval between the steam and the water.

(d) *The Measurement of the Resistance of the Wire Between its Ends.*

Owing to the small temperature coefficient of the alloys used in this research it was sufficiently accurate to find the resistance of a long length of the wire giving an exact balance on an accurate Post Office box, and to measure the length of the experimental wire between its ends, as in Gray's experiments, to about 0·1 mm.

It was found, however, that the resistance of the actual wire could easily be measured in situ by treating it as a low resistance and using the mercury holes a and b of Fig. 1 for the insertion of potential leads. A current of the nominal value 1·018 amperes—which gave a P.D. across a standard ohm. sufficient to balance a Weston cell—could easily be sent through the wire in either direction, while the millivolts across a and b were measured directly on the thermo-electric potentiometer. The good agreement between the two methods is important in showing good contact, definite ends, and smallness of end correction. The balance on the potentiometer was also almost identically the same whether the current was let directly into the wire or in at a terminal on the steam heater except in one case, where the solder was found not to have taken to the copper. It will probably be possible to measure in this way the actual resistance of a wire of normal temperature coefficient under actual temperature gradient provided the mean of the millivolts be taken resulting from the interchange of the potential leads and the reversal of the main current.

By using the same standard ohm in the manners indicated, both as a basis for calibrating the milli-ammeter or measuring C_2-C_1, and also as a means of obtaining a known current for the measurement of R the resistance of the wire, it is noteworthy that its actual value disappears from the calculation for σ and that the accuracy of the potentiometer becomes the important factor.

(e) *The Sensibility of the Balance, the Rapidity of Working, and the Attainment of the Balance.*

When the apparatus of Fig. 1 was fitted with a constantan wire of length 4 cm., supplied with steam and ice so that $U=100$°C. approximately, and was traversed by a current of 3·5 amperes, a full scale deflection of 50 cm. was obtained in just over a minute if the current was merely reversed but not altered in magnitude. Using steam and water at 22°C. and reversing a current of 3 amperes, 30 cm. deflection was obtained in 1 minute and 37 cm. in 2 minutes. With German-silver of the same length, 100°C, interval, and 3·5 amperes 18 cm. deflection was observed in a minute. In manganin the Thomson effect is feeble, viz., 0·68 micro-calorie per coulomb, but for $U=81$°C. a reversal of the same current yielded 5 cm. deviation in the first minute.

This sensibility is very satisfactory, and in further work the author hopes to increase it. Moreover, as the steady state is attained so rapidly it is unnecessary to duplicate by use of a second parallel wire—indeed, such duplication would add difficulties—for, in the short time needed for testing a balance, change of temperature of the detecting coil due to other causes is normally inappreciable. The time of attainment of a new steady state probably shortens on nearing a balance, for in the limit, when the balance is perfect, the middle is a point of stationary temperature and the length of wire is in this respect virtually halved.

Practically the sensibility is not purely a function of the current and temperature difference even for the same wire; for such variable features as the stability of zero of the galvanometer or the steadiness of the sources or "progressive changes" all influence the precision of balance. For example, though U was greater when ice was used instead of ordinary water, the slow progressive rise of temperature of the tank in the former case rendered a balance more difficult and often less precise.

The balance is best approached from both sides by trying first the effect of too big and then of too small a shunt current. After balancing the galvanometer and switching over K_1 and K_2 simultaneously the galvanometer key may again be depressed in about 30 seconds and a movement of the needle looked for. When approaching closely the exact balancing current several reversals should be made so as to make sure that any slight movement of the galvanometer line of light is reversible and not due to a slight progressive change of temperature. Such progressive changes have always caused trouble to experimenters on the Thomson effect. They are due, among other things, to variations in temperature of the hot and cold sources, to insufficient protection of the wire from convection currents, to variation of the main current and the room temperature and to local thermo-electric effects; they may be reduced greatly by special attention to these points and by duplicating the apparatus, but nothing tends more towards their elimination than securing rapidity of attainment of the steady state.

The balance was unaffected if the current instead of entering the steam chamber by the wire under test entered by a terminal on the steam heater.

5. Results of Experiments.

The results of all experiments are tabulated in Table I.

The resistances of the wires used are shown in Table II. The values in the last column obtained by the potentiometer method were those used in the calculation of σ. The resistance values correspond to 20°C. approximately.

In this preliminary work with alloys, the small temperature coefficients were neglected in the calculation of σ the Thomson effect. The best experiments show σ for the constantan used

TABLE I.—*Values of* σ.

Material and number of experiment.		U	Main current amperes.	Balancing current in milliamperes.	Temp. at which σ is found.	$\sigma \times 10^6$ in calories per coulomb.
Constantan A $L = 4{\cdot}0_2$	1	78·5	2·0	88±1	60	$-4{\cdot}7_3$
	2	77·2	2·75	88±1	61	$-4{\cdot}8_1$
	3	76·5	3·5	86·5±0·5	61	$-4{\cdot}7_7$
	4	78·1	2·75	87±1	61	$-4{\cdot}7_0$
	5	78·3	3·5	87·7±0·2	61	$-4{\cdot}7_2$
	6	78·4	3·0	87·5±0·5	61	$-4{\cdot}7_0$
	7	78·0	3·5	87·5±0·5	61	$-4{\cdot}7_3$
	8	99·4	3·0	111±1	50	$-4{\cdot}7_1$
	9	99·5	3·5	111·5±1	50	$-4{\cdot}7_2$
	10	99·6	3·0	111±1	50	$-4{\cdot}7_0$
	11	99·7	3·5	110·5±1	50	$-4{\cdot}6_7$
Constantan B $L = 4{\cdot}4_5$	1	80·0	3·5	80·0±0·5	60	$-4{\cdot}7_0$
	2	79·3	3·0	80·0±1	60	$-4{\cdot}7_4$
	3	99·6	3·0	100±1	50	$-4{\cdot}7_2$
Constantan C $L = 4{\cdot}9_0$	1	79·5	3·5	73·4±0·5	60	$-4{\cdot}6_9$
	2	79·8	3·0	73·4±0·5	60	$-4{\cdot}6_7$
Constantan D $L = 5{\cdot}9_7$	1	83·7	3·5	64·1±0·5	58	$-4{\cdot}7_8$
	2		*See*	Note below		
German Silver G $L = 4{\cdot}0_0$	1	77·8	3·0	87·5±1·5	61	$-2{\cdot}3_3$
	2	78·6	3·5	89·0±1	60	$-2{\cdot}3_5$
	3	99·3	3·0	110·5±1	50	$-2{\cdot}3_1$
	4	99·3	3·5	109±1	50	$-2{\cdot}2_7$
Manganin M $L = 4{\cdot}0_2$	1	80·7	3·5	15·4±0·5	60	$+0{\cdot}6_8$
	2	81·7	3·0	15·4+0·5	60	$+0{\cdot}6_7$

Experiment C_2 was only a retest of balance performed an hour later under 3 amperes. The result of experiment D yielding a higher value was surprising when a lower value was expected; nevertheless, the balance was very definite. Perhaps the rise of the "a" term had more effect than the emissivity loss. On removing nearly all the cotton wool a much lower balancing current—about 60 milliamperes—was required but convection currents then prevent precise measurement.

TABLE II.—*Resistances at 20°C. of Wires Employed.*

Material.	Length in cm.	Resistance calculated from value per unit length	Resistance found on thermoelectric potentiometer.
		ohms.	ohms.
Constantan A......	$4{\cdot}0_2$	$0{\cdot}0173_6$	$0{\cdot}0176_2$
Constantan B......	$4{\cdot}4_5$	$0{\cdot}0192_2$	$0{\cdot}0196_5$
Constantan C ...	$4{\cdot}9_0$	$0{\cdot}0211_6$	$0{\cdot}0212_4$
Constantan D......	$5{\cdot}9_7$	$0{\cdot}0257_8$	$0{\cdot}0261_2$
Manganin M	$4{\cdot}0_2$	$0{\cdot}0147_2$	$0{\cdot}0148_3$
German Silver G	$4{\cdot}0_0$	0·00864	0·00866

to be about $-4{\cdot}7_2 \times 10^{-6}$ calories ($J_{20}=4{\cdot}18$) per coulomb per 1°C. at 60°C. The corresponding values for German silver and manganin are $-2{\cdot}3_5$ and $+0{\cdot}6_8$ micro-calories respectively.

6. Conclusion.

The method appears promising on account of its simplicity, rapidity and sensitiveness, the detector, unlike a thermo-junction, producing no distortion or alteration of electrical resistance at the critical place. The limit of accuracy attainable will probably be found to be associated with thermal and electrical "end-correction." That the Peltier effect has negligible influences is confirmed by König (" Phys. Zeits." 17, June 1, 1916) in a very recent Paper, and additional practical tests of this point are in view by the author. A new apparatus suitable for temperature coefficient work is in the process of design, and the author hopes to carry out the intention expressed in section I. as soon as conditions improve; bismuth wire of the desired gauge is not obtainable at present.

The author must express his thanks to Miss E. M. Humby for valuable mechanical help when difficult to obtain, and especially for services rendered in the successful fixing of the wires through the constant temperature sources. He is also indebted to Dr. A. Griffiths for placing at his disposal for so long a time so much of the best electrical apparatus in the physics department. The expenses of this research were completely defrayed by a grant from the Dixon Fund, University of London, for which the author is very grateful to the Senate.

Summary.

This Paper describes how absolute measurements of the Thomson effect may be made in wires. The theory is fully worked out and the sources of error likely to arise—owing especially to the smallness of the area of cross-section—are considered. The method is sensitive, consistent, and very rapid and the ultimate object is to determine the Thomson effect at different temperatures in a number of metals, both rare and base, at the same time and with the same specimens finding their thermo-electric powers. The preliminary experiments of this Paper, testing the method, are with constantan wires of different lengths, with manganin, and with German silver.

DISCUSSION.

Mr. F. E. **Smith** suggested that the method might have been modified by having two exactly similar systems arranged in opposite arms of a Wheatstone bridge, so that the current passed from hot to cold in one wire and from cold to hot in the other. Any change in the relative temperatures of the two wires on reversing the current would be indicated by the bridge being thrown out of balance. This seemed preferable to measuring the temperature of a wire by the resistance of another wire wound round it.

Mr. D. **Owen** thought the method was very satisfactory. Had the author thought of using electrical heating for the hot end in order to get smaller temperature gradients, in view of the high sensibility of the arrangement? He thought the figures of Oosterhuis stood in need of verification. If the author could adapt his method to deal with crystals, such as were employed in wireless detectors, the results would be of very great interest.

Dr. R. S. **Willows** said that it was difficult to see from the equations which Mr. Nettleton had put down on the board whence the energy was derived in cases such as those mentioned in the Paper which Mr. Darling was about to read. In some of these cases dE/dT and d^2E/dT^2 were both zero. It would be useful if the method could be rendered applicable to substances with large temperature coefficients.

Prof. W. **Eccles** said he had hoped that the author would have explained just what the important points of the method were in a less abstract manner—with the aid, say, of diagrams of the temperature conditions along the wire before and after the current reversal.

The **Author**, in a written reply, thanks Mr. F. E. Smith for his suggestion, which, however, as regards sensibility, would depend on the value of the temperature coefficient of resistance, and might carry with it additional complications such as those which enter into King's experiments. In reply to Mr. Owen, the use of electrical heaters would simplify temperature coefficient work enormously, but unfortunately there are mechanical difficulties arising out of the facts that the solder fixing the wire must be applied from behind, and that, in order that the resistance of the wire may be measured as a low resistance in situ, the copper must be narrowed down, as shown at K in Fig. 2, leaving a very small bearing surface at N. The author still hopes that he may be able to overcome these difficulties, while ensuring at the same time that the difference of temperature U, measured by the thermocouple, is substantially that between the extremities of the wire. He agrees with Mr. Owen that the results of Oosterhuis should be confirmed, as well as those of Aalderink, in view of the extraordinary conclusions to which they lead; at the same time, he has a very high opinion of the methods adopted at Groningue, and his own results on mercury some time ago were not very different from those of Schoute In reply to Prof. Eccles, the author has himself felt the difficulty of explaining the essence of the method simply and satisfactorily. Graphs of the distribution of temperature afford little help. What is actually done is explained in a few words in section 2 of the Paper, though no simple proof of the formula is given. A "proof" of the formula $\sigma=R(C_2-C_1)/SU$ is obtained very simply by equating the electrical heat produced per second per centimetre—viz., $C_1^2R/SL+C_1\sigma U/L$, when C_1 is passing from hot to cold, to that produced when C_2 is passing in the opposite direction—viz., $C_2^2R/SL-C_2\sigma\cdot U/L$. This proof was omitted not so much because it assumes that σ, R and the temperature gradient are constant everywhere, and ignores emissivity, as because it neglects thermal conductivity, or assumes that Joulean and Thomson heat are propagated similarly. In reply to Dr. Willows, the method is equally applicable to wires of large temperature coefficient. Equation (2B) may then be applied, or equation (2C) if the resistance of the wire be measured directly while under temperature gradient, as suggested in section 4D. The author has no satisfactory explanation of the energy problems arising out of the researches in Holland, or those arising from Mr. Darling's experiments: in the absence of measurements of σ and π there are few data to go upon. In the case of Mr. Darling's experiments, it must be remembered that though with, say, the bismuth silver couple over the range 300°C.-500°C., both dE/dt and d^2E/dT^2 are zero, no additional energy or thermo-electric force is manifest. If the law of successive temperatures is true with Mr. Darling's materials, a thermocouple composed of molten bismuth and silver with one junction at 300°C. and the other at 400°C. or 500°C., should show no E.M.F., corresponding to zero Peltier effects and equality of Thomson coefficients.

V.—*On the Thermo-electric Properties of Fused Metals.* By CHARLES ROBERT DARLING, *A.R.C.S.*, *F.I.C.*, *and* ARTHUR W. GRACE.

RECEIVED OCTOBER 30, 1916.

1. *Bismuth.*

IN measuring high temperatures by the thermo-electric method, it has hitherto been customary to regard the melting point of either member of the couple as the extreme temperature capable of being measured. As pyrometers of this type are now constructed, the fusion of either metal causes a break in the circuit, and hence for temperatures exceeding 1,000°C. it has been found necessary to use junctions of platinum and an alloy of platinum and rhodium. If, however, a pyrometer could be arranged so that the fusion of either member of the junction would not destroy the circuit, it might be possible to use cheap metals over very high ranges of temperature, provided the boiling points were not too nearly approached. The boiling points of metals have been investigated by Greenwood, who has shown that the values for common metals frequently exceed 2,000°C. ; tin, for example, boiling at 2,270 deg., copper 2,310 deg., and iron 2,450 deg. It would, therefore, appear feasible, in a specially designed pyrometer, to measure temperatures as high as 1,500 deg., and probably higher, by using common metals in the molten condition, provided their thermo-electric properties were suitable. Further, in the case of bismuth, which boils at 1,420 deg., the high E.M.F. developed when this metal is joined to iron, copper, &c., suggests the possibility of delicate measurements over a range from 0 deg. to 800 deg. or 1,000 deg., again assuming that the thermo-electric properties after fusion are satisfactory.

With these objects in view an investigation was commenced by the first-named author about three years ago, but owing to the abnormal conditions prevailing, progress has been very slow. As a preliminary, it was necessary to determine whether any notable change in thermo-electric properties was caused by fusion, and the results arrived at in this connection were given at the meeting of the Society held on June 30th, 1916. It was then shown that in the cases of lead, tin, zinc and cadmium, using a variety of metals as second members of the junctions, that no perceptible break occurred in the continuity of the curves obtained by plotting E.M.F. against temperature, either at fusion or at any subsequent stage up to 700°C. With bismuth, however, several cases were noted in which the E.M.F. remained nearly constant after fusion over a wide range of temperature, reproducing the phenomenon of thermo-electric "halt" observed by Barrett* with a junction of iron and an alloy of iron, nickel and manganese containing 1·2 per cent. of carbon. This property of molten bismuth has since been more carefully examined, and also the behaviour of molten bismuth in cases where this halt is not obtained; and the present Paper is devoted to the special case of bismuth. Work is being carried on, as circumstances permit, with a view to accomplishing the main object of the research, and it is hoped to communicate the results at a later date.

Historical.

The effect of fusion in the case of an iron-bismuth junction was first observed by von Obermayer in 1872†. The bismuth was placed in a funnel provided with a long stem, and a junction formed at the upper and lower end, and the upper layer of bismuth in the conical part was melted by heating this portion of the funnel. A complicated method for measuring the thermo-electric change was used, in which values were referred to a copper-german silver element. A falling-off in E.M.F. was noted after fusion.

Further observations on similar lines were made by Minarelli in 1875,‡ who found that with junctions of Bi-Fe and Bi-Pt the E.M.F. diminished after fusion, whereas with Bi-Cu the effect of fusion was to slow down the rate of increase. Observations were made up to 380°C. with Bi-Fe and Bi-Cu; and up to 500 deg. with Bi-Pt. The results obtained suffered from the absence of an accurate method of measuring the E.M.F. developed.

No systematic examination of the thermo-electric properties of fused bismuth appears to have been made since this date, although one or two isolated results have been recorded bearing on the effect of fusion, but not upon the properties of the liquid metal at temperatures much higher than its melting point (269°C.). Thus K. Siebel§ found that with a manganin-bismuth junction the E.M.F. temperature curve changed in direction at the melting point, and became horizontal up to 300°C., beyond which it was not traced. He observed also a change at the melting point of an alloy of antimony and cadmium when joined with constantan. The results embodied in the present Paper include measurements of E.M.F. against temperature up to 560°C. with junctions of bismuth and a variety of metals, and are thus an extension of previous work.

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* "Phil. Mag.," Vol. XLIX., p. 309, March, 1900.

† "Sitzb. der k. Akad. der Wissench.," Vol. LXVI., 1872.

‡ "Sitzb. der k. Akrad. der Wissench.," Vol. LXXI., 1875.

§ "Annalen der Physik," Vol. XLV., 6, pp. 839-860, 1914.

Experimental.

In order to maintain the circuit after fusion, a salamander crucible, *A* (Fig. 1), was inserted in a sheet of uralite so as to leave the upper rim projecting about 0·5 cm. A groove was cut in the uralite, and a corresponding groove filed in the crucible, so that the molten metal could overflow into the channel in the uralite. This channel was filled with bismuth to a distance of about 50 cm. from the crucible, where a cold junction, *C*, was formed with the metal under test, which was fused into the bismuth. The hot junction, *H*, was made by dipping the metal directly into the crucible, or, in cases where the metal was attacked by the bismuth, by surrounding the wire by a tight graphite sheath which was immersed in the crucible. Temperatures were measured by means of a carefully calibrated iron-constantan junction, protected by a graphite tip, and immersed in the bismuth side-by-side with the hot junction. Some of the earlier results were checked with a pressure thermometer, which was broken and could not be replaced. The solidification of the bismuth served as a check to the pyrometer in each experiment, and in no case was a greater discrepancy than 2 deg. observed. The temperature of the cold junction was taken from a thermometer placed with its bulb touching the junction, and varied between 18 deg. and 20 deg. on different days, not being noticeably affected by the flow of heat along the bismuth. The crucible was heated by a Méker burner, which proved capable of raising the tempera-

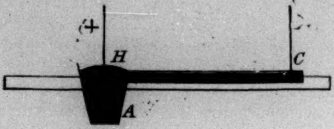

Fig. 1.

ture to nearly 600°C. E.M.F.s were measured directly on a potentiometer made by the Cambridge Scientific Instrument Company, this instrument being specially suited for thermoelectric work. A Weston cadmium cell was used as standard, and two sets of cooling readings were taken with each junction, which generally showed a good agreement, frequently to within 0·1 mv. In drawing the temperature-E.M.F. curves average values were taken to the nearest $\frac{1}{20}$ mv. Owing to the long time taken in cooling, observations below 100 deg. were not usually made, as such did not bear on the object of the investigation. The results obtained are shown in the following Table :—

Temp. of hot junction.	E.M.F. in millivolts, Bismuth with										
	Copper.	Iron.	Aluminium	Nickel.	Brass.	Silver.	Constantan ?	German Silver.	Nichrom.	Graphite.	Platinum.
104°C.	5·75	6·40	5·70	4·35	5·60	6·55	4·25	4·85	5·65	6·15	5·50
126	7·50	8·10	7·15	5·60	7·35	8·30	5·40	6·15	7·35	7·85	7·05
148	9·00	9·60	8·75	6·65	8·90	10·05	6·50	7·50	8·90	9·50	8·45
172	10·60	11·50	10·05	7·90	10·40	11·70	7·65	8·75	10·70	11·40	10·00
192	12·00	13·00	11·75	8·85	11·95	13·50	8·55	9·95	12·10	12·90	11·40
216	13·60	14·25	12·75	9·70	13·30	15·00	9·45	10·90	13·55	14·30	12·70
238	14·80	15·30	13·60	10·40	14·55	16·25	10·05	11·70	14·75	15·70	13·70
260	15·70	16.30	14·65	10·75	15·45	17·25	10·40	12·20	15·80	16·65	14·45
269 m.pt.	15·75	16·30	14·65	10·75	15·75	17·65	10·45	12·30	15·80	16·65	14·90
284	15·95	16·40	14·70	10·15	15·70	17·45	9·85	11·95	15·80	16·90	14·70
305	16·00	16·45	14·70	9·60	15·60	17·35	9·45	11·45	15·80	16·80	14·55
328	16·10	16·50	14·65	9·00	15·50	17·30	8·90	10·90	15·75	16·70	14·40
352	16·20	16·50	14·65	8·50	15·35	17·30	8·35	10·25	15·75	16·55	14·25
372	16·35	16·50	14·70	7·95	15·30	17·25	7·55	9·75	15·75	16·45	14·05
396	16·40	16·50	14·90	7·55	15·25	17·30	6·80	9·15	15·75	16·40	13·85
418	16·60	16·40	14·90	7·20	15·25	17·30	6·10	8·40	15·75	16·30	13·70
440	16·75	16·40	14·95	6·80	15·25	17·25	5·20	7·80	15·85	16·20	13·50
464	16·85	16·40	15·00	6·30	15·35	17·20	4·45	7·40	15·90	16·05	13·30
486	16·95	16·50	15·00	5·90	15·40	17·10	3·65	6·80	16·05	15·90	13·15
508	17·10	16·60	14·65	5·60	15·50	17·15	2·95	6·20	16·20	15·75	12·90
532	17·20	16·65	14·75	5·30	15·60	17·30	2·55	5·75	16·35	15·65	12·80
553	17·25	16·60	14·75	4·85	15·70	17·35	1·50	5·10	16·40	15·50	12·65

Cold junction, 18° to 20°.

Interpretation of Results.

The results obtained show in all cases a marked change in thermo-electric properties at the melting point of bismuth, the nature of the change, however, depending on the second metal. Fig. 2 represents the temperature-E.M.F. curves of copper, silver and nickel, and it will be noticed that at the melting point of the bismuth an abrupt change occurs in each instance. In the case of copper, a small but steady increase in E.M.F. is observed after fusion; with silver the E.M.F. is practically stationary between 300 deg. and 500 deg.; whereas a rapid fall in E.M.F. occurs with nickel. The other metals experimented with give curves resembling one or other of the typical examples shown. Brass shows a slight fall in E.M.F. between 280 deg. and 350 deg., after which a slow, steady rise is observed. Iron, aluminium and nichrom resemble silver in producing a stationary or nearly stationary E.M.F. with the fused bismuth.

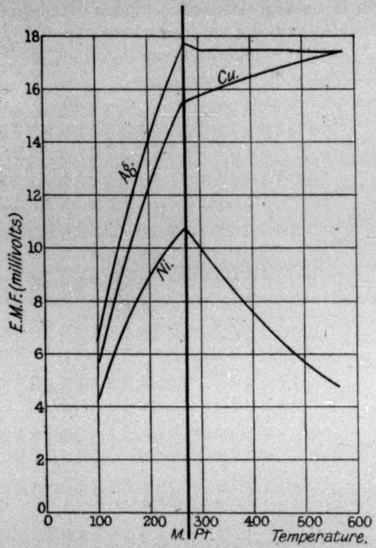

FIG. 2

German silver and constantan give curves resembling that of nickel; whilst graphite and platinum show a more gradual but continuous fall in E.M.F. after the bismuth has melted. The sample of constantan wire was purchased as such, but did not give so large an E.M.F. either with solid bismuth or iron as is generally attributed to this alloy, and was probably different in composition to the ordinary material.

No attempt was made to construct a thermo-electric diagram from the results—as it would be impossible to obtain correct values of dE/dt at all parts of the curves, many of which show small flexures—without making measurements of a much higher order of accuracy in the region of these flexures. This would involve temperature measurements of a greater degree of refinement than is possible with the methods employed; and it is doubtful whether any conclusions of value could be deduced from such a diagram even after specially accurate results had been obtained. When the temperature-E.M.F. curves are parabolic, and thermo-electric powers may be represented by a straight line, the diagram may be of service; but as this does not apply to the greater part of our results no attempt in this direction has been made. We may point out here that the diagram originally given by Tait, and still reproduced in books is in error, as showing a neutral point in the case of iron and lead between 350 deg. and 400 deg., followed by a diminishing E.M.F. Using the experimental methods described in the present Paper, we find the E.M.F. of a lead-iron junction rises continuously up to 650 deg., and that the neutral point supposed to exist about 40 deg. above the melting point of lead is purely imaginary, having been arrived at by unjustified extrapolation. The usual form of thermo-electric diagram, consisting of a number of straight lines crossing at various points, is of interest mathematically, but is quite unsafe as a guide to the thermo-electric properties of metals over long temperature ranges, and particularly so when extrapolated beyond the fusion points.

The nearly constant E.M.F. observed with silver, aluminium, iron and nichrome over a range of nearly 200 deg. beyond the melting point of bismuth suggests the possibility of a steady thermal source of E.M.F., the value of which would be approximately 17, 15, 16·5 and 16 mv., taking the metals in the order given above. An E.M.F. of this order of magnitude is used in thermo-electric pyrometers in opposition to the hot junction, so as to enable the whole scale to be used for a given range of temperature—say, 500 deg. to 1,000 deg.—the opposing E.M.F. preventing the movement of the indicator until overcome by that of the hot junction. At present a cell, broken down by resistances is employed to furnish this counter E.M.F., repeated adjustment by reference to a standard cell being necessary as the working cell runs down. A thermal cell, if durable, and sufficiently accurate, would be cheaper and simpler in use than the method now employed, and experiments have been carried out with a view to testing the practicability of a thermal cell for this and similar purposes. It is quite simple to adjust a gas flame so that the temperature of the molten bismuth does not exceed an assigned limit, such as 400 deg., and practical permanence of the temperature of the cold junction can be secured by surrounding it with water. In order to discover whether repeated heating caused any change in the E.M.F., an Al-Bi junction was heated for a total of 70 hours without any alteration being detected; and in no case has prolonged heating been observed to alter the E.M.F. A difficulty, not yet overcome, is the oxidation of the bismuth, which takes place to such an extent as ultimately to break the circuit. Work is still in progress with a view to finding which of the four metals named is the best for practical service, and to secure permanence by preventing oxidation. One drawback to the use of aluminium is that if heated above 370 deg. the E.M.F. rises in value from 14·7 to 15·0 mv. at 460 deg, further heating causing the value to fall to 14·75 at 530 deg.; and hence 370 deg. is the highest permissible temperature. This leaves a range of only 100 deg. over which the E.M.F. is constant to within 0·05 mv., and would necessitate careful regulation of the flame. With silver, the corresponding range is between 320 deg. and 440 deg.; with iron, 300 deg.-400 deg.; and with nichrom 270 deg.-420 deg. The last-named appears to be the most promising, but no final conclusion can be arrived at until prolonged tests have been made. For any temporary purpose, however, an arrangement such as that described in Fig. 1 may be used as a steady source of low E.M.F., using nichrom wire and a flame just sufficiently high to keep the bismuth melted. This will be found useful in measuring low E.M.F., evaluating galvanometer deflections and similar operations.

It is evident, from the results, that molten bismuth cannot be utilised for the measurement of temperatures. Its thermoelectric properties, after fusion, are different from those of tin, lead, zinc and cadmium, all of which show no marked discontinuity as the result of fusion. Preliminary experiments with antimony suggest a similarity to bismuth in this respect, but owing to the higher melting point of the former (632°C.) the apparatus used with bismuth is unsuitable. An electric heating arrangement is under construction with which it is hoped to attain temperatures of 1,000 deg., in order that the properties of various fused metals may be examined over a wider range.

The present investigation has been carried out in the Heat Laboratory of the City and Guilds Technical College, Finsbury, E.C.

ABSTRACT.

One of the authors has for some time been investigating the possibility of using base metal thermocouples at temperatures above the melting point of one of the constituents. For this purpose it was necessary to determine whether any peculiarities in the thermoelectric behaviour of metals occur at fusion. In the case of lead, tin, zinc and cadmium there is no perceptible break in the continuity of the curves obtained. In couples containing bismuth, however, several cases were noted in which the E.M.F. remained constant for a wide range of temperature after the fusion of the bismuth. This occurs with silver, aluminium, iron or nichrom as the other element. Useful applications of this property are discussed. No attempt was made to construct a thermo-electric diagram from the results, as it would be impossible to obtain correct values of dE/dt at all parts of the curves, many of which show small flexures, without making more accurate measurements in the region of such flexures than was practicable.

DISCUSSION.

Mr. A. Campbell, in a communication which was read by the Secretary, remarked that he had made similar experiments (Proc. Roy. Soc., Edin., 1888) on Wood's metal, an alloy of bismuth melting at 73°C. The results of measurements against iron from 8°C. to 150°C., when reduced to a thermo-electric diagram, gave one straight line up to the melting point, and another straight line, of a different slope, above the melting point. He disagreed entirely with the authors' remark son the usual thermo-electric diagram. A diagram of the thermo-electric powers was most valuable, even though the lines may not be straight at the extreme temperatures. If anything is to be deduced from observed results, the first step is to plot the thermo-electric powers against temperature. Why the authors give results of tests against a whole series of metals is not clear. A set of tests against one metal would have been sufficient to fix the bismuth curve on the thermo-electric diagram, and all the other results could then have been deduced from the curves of the other metals already determined by more exact methods.

The President expressed himself puzzled by the results shown in Fig. 2. Up to the melting point of bismuth the different metals gave with it increasing E.M.F.s. At the melting point, however, discontinuities occurred, and it appeared that if one eliminated bismuth by subtracting the ordinates of one curve from another, discontinuities would also occur, at the melting point of bismuth, in the properties of the couples obtained by combining the other metals *inter alia*. This, of course, was not true.

Prof. **Howe** pointed out that there was not a marked discontinuity in the difference of the nickel and copper curves when this was plotted. The President's remarks held, however, in the case of silver with either of the others.

Mr. S. W. **Smith** pointed out that bismuth was abnormal in that it contracts on melting.

Dr. R. S. **Willows** mentioned that the metals lead, tin, zinc and cadmium, which had been shown to have no thermo-electric discontinuities at their melting points, happened to be those which, as Mathiessen had found, formed alloys with one another of which the specific resistance could be deduced from the percentage composition. He suggested that the discrepancy mentioned in the case of iron might possibly be due to the occurrence of some allotropic modification, in which case there was really a change in the substance, and it was unfair to extrapolate the thermo-electric diagram beyond the change point.

Mr. **Darling**, in reply, said he had no doubt that Mr. Campbell's remarks would hold if one could get out the proper thermo-electric diagram. Owing to the numerous little kinks which occur, very sensitive temperature measurements are required. If really accurate diagrams could be obtained it might then be possible to deduce the properties of any bismuth couple from those of one, as Mr. Campbell suggested. They had tried this in a few cases, however, but did not get good agreement. He did not know whether the peculiarity of bismuth which had been mentioned was related to the phenomena which they had described. Bismuth was a freak metal in many respects. He was unable at the moment to solve the difficulty raised by the President. Their aim had been, primarily, a practical one, and they had not gone as fully into points of that nature as might have been desired.

VI. *On the Regularity in the Distribution of the Satellites of Spectrum Lines.* *By* PROF. H. NAGAOKA, *Imperial University, Tokyo.*

RECEIVED SEPTEMBER 4, 1916.

§ 1. The nature of the satellites of spectrum lines is not yet thoroughly known, nor are the positions with respect to the principal well determined, except for a few lines of a small number of elements. I have no doubt that there exists regularity in the distribution of the satellites, in their order of intensity, and in their behaviour when acted upon by strong electric and magnetic forces. The usual view as regards the satellites is to consider them as belonging to the principal line, which is the strongest among an assemblage of lines of various strengths clustered together within a narrow compass about the principal. It is still an open question whether such grouping of lines is swayed by the principal, or they have independent existence apart from the principal. There is as yet no observation of the effect of an electric field on the satellites, and on account of the difficulty of measurement and of the complexity of the separation the Zeeman effect is known only for a limited number of satellites. Experiments on the effect of the magnetic field on the satellites of mercury lines show that some of them are controlled by the principal or strong satellites in the neighbourhood, but there exist satellites which seem to lead an independent existence, the mode of magnetic separation being little affected by the principal. It may, therefore, be improper to call them satellites, while some of them are strongly influenced and bear the character of the name assigned to them.

§ 2. In a former communication to the Society,* we have noticed that the satellites of mercury lines are apparently distributed with some regularity and that there is gradation in intensity. Although no definite result could then be obtained, it was probable that the minute structure of the principal lines was not yet known, as most of the discrepancies among different observations were to be traced to the inaccuracy in the determination of the exact position of the principal lines. As suggested by Stansfield, the position of the satellites referred to one of the distinct components gave tolerably good concordance in the results of different observers with various instruments, and it seemed quite likely that further study of the principal line would reveal the regularity in the distribution of the satellites, if such really exists.

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* Nagaoka and Takamine, "Proc." Phy. Soc., 25, p. 1, 1912.

§ 3. For this purpose, it is necessary to use instruments with high resolving power, in order that a strong line with rather vague boundaries may be separated into its components. A concave grating is generally inefficient, the echelon grating and Lummer-Gehrcke plate have definite resolving power for a given wave-length, so that we can not bring different powers to bear upon the lines to be analysed. The inconvenience is especially felt when the interval between successive spectra is so narrow that the lines are superposed and cause great confusion in discriminating them. The disadvantage arising from the definiteness of the resolving power can in no wise be remedied, so that it is generally expedient to resort to those interferometers in which the thickness of the air space contributing to interference can be easily changed, so as to suit the purpose of the experiment. The interferometers of Michelson and of Fabry-Perot belong to this type of instruments, as the air interspace is adjustable within wide limits. A simple device for changing the resolving power is a prime factor that makes the instrument indispensable in researches relating to the minute structure of a spectrum line, especially when it consists of a cluster of lines within narrow limits. The easiest way of arriving at a desired resolving power would be to increase the air interspace between the plates in a Fabry-Perot interferometer, the thickness being, of course, within the limit of interference for the line to be examined. The exact value of the resolving power cannot, however, be given, as it depends on the number of reflections from the semi-transparent silver films. One serious objection to a thick interspace is the crowding of the lines, as different interference fringes are intermixed with each other, and hence arises the trouble of discriminating the different lines, especially when there are numerous satellites. In fact, the appearance of interference rings is extremely vague, when the interspace is several centimetres thick, that it is almost impossible to tell even the position of the strong satellites. This difficulty can, however, be to a great extent overcome by the use of interference points, obtained by crossing it with another instrument, which may have either fixed or adjustable resolving power. The advantage* gained by crossing is especially great, when there is ambiguity in the position of the satellites as observed with a single instrument; for it can easily be settled by examining the crossed spectra. With an adjustable interferometer, we can change the air interspace, such that the satellites to be analysed are brought distinctly under observation, without being intermixed with other components. I have tried the combinations of F.P. interferometer with echelon grating, L.G. plate, wedge plate, and another F.P. interferometer of the fixed type; the last method of crossing was found to have a disadvantage of losing too much light in passing through four semi-transparent silver films, so that most of the measurements were made with the two other combinations. In addition to this, I have found that the combination of echelon grating or L.G. plate with a wedge plate forms an excellent pair for work with fixed resolving power.

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* Nagaoka and Takamine, "Phil. Mag.," 27, p. 126, 1914.

§ 4. Preliminary study of the minute structure of mercury lines was made in conjunction with Mr. Takamine,* and published in connection with the Zeeman effect of the mercury line λ 4,359; but subsequent experiments showed that there were many points still to be improved in the method of investigation, especially in increasing the order of interference, for making the fine structure of the principal line easily visible on the photograms.

§ 5. It was first shown by Janicki† that the principal line of the green line of mercury is multiple, by combining wedge and L.G. plates, which had a resolving power of 680,000 for the said line. Although Janicki denies the multiple character of other lines of mercury, it seemed quite likely that fine structure of the principal lines could not be placed under a strict experimental test, inasmuch as the resolving power of his instrument was fixed. With sliding F.P. interferometer, the resolving power can be raised to several times the above value, and the structure more accurately analysed.

§ 6. In the present experiment, the air interspace was much greater than in the former and generally ranged between 6 cm. and 12 cm.; with the thickness of 7 cm. or 8 cm., the composite character of the principal of the green line was distinctly visible, but owing to the overlapping of different spectra, it was necessary to cross it with a L.G. plate whose $d\lambda_{max.}=0{\cdot}1066$ A.U.; the resolving power thus utilised may be estimated at $2{\cdot}3\times 10^6$ for 8 cm. air space, assuming the number of reflections on the front face to be 8. The spectra were drawn out more than with the echelon grating; but, owing to the crowding of interference points resulting from numerous satellites, it was somewhat difficult to discriminate them; from our previous knowledge of the position of the satellites, we can, however, avoid their coincidence with the principal line, by calculating the value of the air interspace suitable for the experiment. In addition to this, the intensity of the principal line was far greater than that of the satellites, so that there was no trouble in following the curves of interference arising from the components of the principal line.

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* Nagaoka and Takamine, "Phil. Mag.," 29, p. 241, 1915.

† Janicki, "Ann. d. Phys.," 33, p. 438, 1912.
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§ 7. The mercury lamp was of the Arons-Lummer type, water-cooled, and run with a current of 2·5 to 3 amperes at 100 volts; in the circuit was included a big electromagnet having sufficient inductance to maintain the current steadily. Great care was taken to keep the temperature of the air interspace constant; the resistance and other parts, which are liable to be heated, were kept outside the room. The interferometer was enclosed in a wooden box, leaving only a passage for light and photographic apparatus. Wratten-Wainright panchromatic plates were generally used for the green and yellow lines, and process plates for the violet lines. The interference points were measured by means of a micrometer made by Repsold, or with a dividing engine constructed by Société Genèvoise, having attached microscopes and an arrangement for eliminating screw error.

§ 8. It will be necessary to give a short explanation of the locus of interference points obtained by the combination of interferometers above mentioned. For the F.P. instrument, the order of interference P for light of wave-length λ subtending a ring of small angle 2θ is

$$P=\frac{2e\cos\theta}{\lambda}=\frac{2e}{\lambda}\left(1-\frac{\theta^2}{2}\right),$$

e denoting the thickness of the air interspace. Referred to rectangular co-ordinates x and y through the centre of the ring in the focal plane of the observing telescope, and supposing that the wave-length λ is different by a small amount $\delta\lambda$ from λ_0, we get

$$\frac{x^2+y^2}{2}=1-\frac{P}{2e}(\lambda_0+\delta\lambda)=\left(1-\frac{P}{P_0}\right)-\frac{P}{P_0}\,\frac{\delta\lambda}{\lambda_0},$$

where P_0 stands for $2e/\lambda_0$.

Since P is nearly equal to P_0 and very large,

$$\frac{x^2+y^2}{2}=\frac{\delta P}{P_0}-\frac{\delta\lambda}{\lambda},$$

where $\delta P=P_0-P$.

The order h for L.G. plate is given by

$$h=\frac{2t\sqrt{(\mu^2-\sin^2 i)}}{\lambda},$$

where t is the thickness, μ the index of refraction for light of wave-length λ, and i the angle of exit. As observations are generally made near grazing exit, we can put

$$i=90^\circ-\alpha$$

where α is small. Thus,

$$h^2\lambda^2=t^2\{4(\mu^2-1)+2\alpha^2\}.$$

Putting $\lambda=\lambda_0+\delta\lambda$, where λ_0 is the wave-length of the reference line, and $\delta\lambda$ is the difference in wave-length of the satellites from it, we get

$$2h^2\lambda_0\delta\lambda=\{4t^2(\mu^2-1)-h^2\lambda^2\}+2t^2\alpha^2,$$

the quantity within the parenthesis being constant.

§ 9. On combining the plate with F.P. interferometer, the centre of the ring does not always coincide with the direction of grazing incidence; we have, therefore, to put $\alpha=a+x$; consequently $2h^2\lambda_0\delta\lambda=2h_0{}^2\lambda_0\delta\lambda+2t^2(a+x)^2$ nearly.

Equating $\delta\lambda$ in these two equations for F.P. and L.G. interferometers, and using the relation $\frac{4t^2}{h_0{}^2\lambda_0{}^2}=\frac{1}{\mu^2-1}$, we get the equation

$$x^2+\frac{(a+x)^2}{2(\mu^2-1)}+y^2=2\left(\frac{\delta P}{P_0}-\frac{\delta h}{h_0}\right),$$

which is evidently an equation to an ellipse. Thus the interference points to be obtained by the said combination lie on ellipses with the centre on the x-axis. If we now adjust the position of the plate, such that the ray at grazing exit passes through the centre of the rings, $a=0$, and the equation reduces to

$$\frac{2\mu^2-1}{2(\mu^2-1)}x^2+y^2=2\left(\frac{\delta P}{P_0}-\frac{\delta h}{h_0}\right).$$

The major axis of the ellipse is evidently on the y-axis, and the excentricity $=1/\sqrt{(2\mu^2-1)}$. In the present experiment for λ 5,461, the eccentricity amounts to 1/1·994 and the ratio of the axes 1·156, being nearly $=\sqrt{4/3}$. For the same value of δP, the axes of the ellipse become smaller as the air interspace is increased, so that the ellipses for successive orders of P become denser, till they are obliterated by coming into contact, as the limit of interference is reached. With the green and the violet line of mercury, the interference is still visible for $e=12$ cm., corresponding to the values of $P=440,000$ and $550,000$ respectively. From the measurements of the breadth of interference points, I found the order at the limit of interference to be 750,000 and 810,000 respectively, nearly coinciding with the values found by Fabry and Perot by another method.

§ 10. For a given wave-length, the value of h in the L.G. plate is fixed, as the angular interval of the successive spectra cannot be changed. With F.P. interferometer, this is nevertheless the case. With increased air space, the interval for the same value of P decreases, as the angular difference for successive spectra is given by the relation $\Delta\theta=-\frac{\lambda}{2e\theta}=-\frac{1}{P_0\theta}$. Consequently the value of $\frac{\delta P}{P_0}-\frac{\delta h}{h_0}$ becomes smaller as the interspace is widened. As the axes of the ellipse on which the interference points for the same orders of P and h lie are determined by the above term, the curve for the same difference of P and h occupies a smaller space as the resolving power of the interferometer is increased. The experiments in this direction indicate that the disadvantage thus caused is of little consequence, provided the temperature of the interspace does not vary during observation or photographing.

§ 11. Another important question in connection with the present combination is the variation of x and y co-ordinates, as we pass from one interference spectrum to that succeeding it. For simplicity's sake, I shall make the supposition that the ray at grazing incidence passes through the centre of the rings, *i.e.*, $a=0$. Keeping λ constant,

$$h^2-h_0^2=\frac{2t^2}{\lambda^2}x^2$$

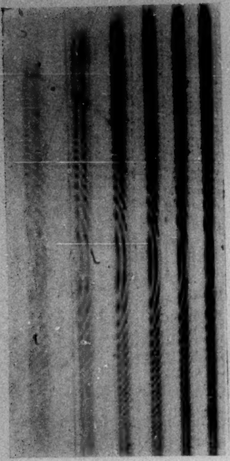
for L.G. plate and
$$P-P_0=-\frac{e}{\lambda}(x^2+y^2)$$

for F.P. interferometer. For finite differences of h and P, we get approximately

$$\lambda^2 h\Delta h=2t^2x\Delta x$$

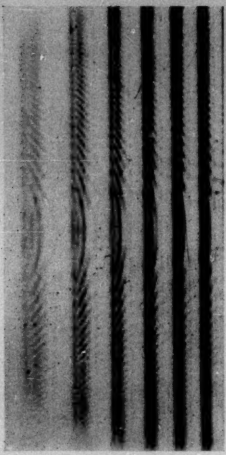
and
$$\lambda\Delta P=-2e(x\Delta x+y\Delta y).$$

Fig. 1a.
λ 5,461.



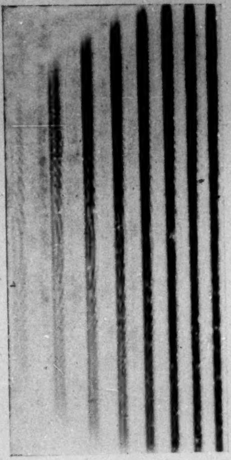
e=8·8 *cm.*
L.G.+F.P.

Fig. 1b.
λ 5,461.



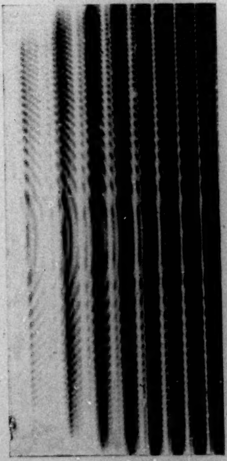
e=8·0 *cm.*
L.G.+F.P.

Fig. 1c.
λ 5,461.



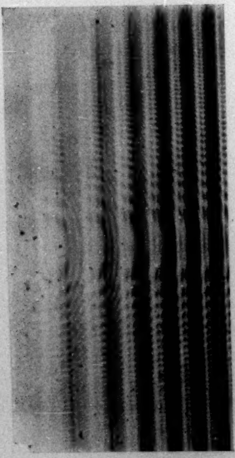
e=9·5 *cm.*
L.G.+F.P.

Fig. 3a.
λ 4,359.



e=5·0 *cm.*
L.G.+F.P.

Fig. 3b.
λ 4,359.



e=6 *cm.*
L.G.+F.P.

To face p. 97.

In these combinations, let us pass from P to $P+\Delta P$ by keeping h constant, then

$$\Delta x=0$$

and

$$\Delta y=-\frac{\lambda}{2ey}\Delta P=-\frac{1}{P_0 y}\Delta P,$$

giving the change in y. Next vary h and keep P constant; then x and y are to be found from the above two equations by putting $\Delta P=0$; evidently

$$\Delta x=\frac{\lambda^2 h}{2t^2 x}\Delta h=\frac{2(\mu^2-1)}{x}\frac{\Delta h}{h}$$

$$\Delta y=-\frac{\lambda^2 h}{2t^2 y}\Delta h=-\frac{2(\mu^2-1)}{y}\frac{\Delta h}{h}$$

approximately.

These simple relations will help us to understand how the curves of interference points are crowded, as we pass outwards from the centre of the rings, showing that the variations of the co-ordinates for successive spectra are inversely proportional to them.

§ 12. From the above discussion, it will be seen that for calculating the difference in wave-length of interference points referred to the L.G. plate, there is great inconvenience in the gradual decrease in the intervals between the successive spectra as the order h increases from that at grazing exit. Interpolation by simple proportional parts is inadmissible, so that we have to take the second differential into account. This is a rather tedious process, so that for interpolation, the graphical method is perhaps the easiest and the least liable to error. This process was followed in evaluating the readings of the micrometer.

§ 13. Figs. 1 (A, B, C) show the photograms of the line λ 5,461 obtained by the above-mentioned combination for three values of e. The principal line is evidently a triplet with a strong satellite on the positive and negative sides. These points are connected by deep shades, as the resolving power of the plate is not sufficiently high; it is on account of the high resolution of the F.P. interferometer that these points appear distinctly. Several photographs for different values of the air space were taken, all leading to the same conclusion; their distances are as folows :—

$-19{\cdot}9 \quad -8{\cdot}4 \quad 0 \quad +8{\cdot}2 \quad +18{\cdot}4$ m.Å.U.;

the remaining points are due to satellites, of which the outermost one -237 is conspicuous, and may be used as a reference line for comparing the results of different observers, as suggested by Stansfield. For finding the position of other satellites the air space of about 1 cm. is preferable to that above mentioned. The following table will show how the results can be reconciled with each other, if the lines be referred to the said satellite, and not to the principal line whose position is very vague in ordinary experiment, owing to its multiple character. The last column gives the difference in wave-number per centimetre referred to the principal line :—

Von Baeyer.	Nagaoka & Takamine.		Janicki.	Nagaoka.	
L.G. + L.G.	Echel. + L.G.		L.G. + Wedge	F.P. + L.G.	$d\nu$.
0	0	0	0	0	+0·793
136	133	134	134	135	+0·341
169	166	168	168	167	+0·234
188	191	188	188	190	+0·157
213	220	216	214	216·9	+0·066
...	...	...	227	228·4	+0·027
237 ?	242 ?	246 ?	236	236·8	0
...	...	...	245	245·0	–0·029
...	...	...	254	255·2	–0·062
319	318	320	320	322	–0·285
362	364	365	364	365	–0·431
448	450	452	450	451	–0·717

The above results were all obtained by crossing the two instruments mentioned under the names of the observers ; the comparison shows that the position of the satellites and their number are now well settled for the green line.* The mutual agreement of these numbers affords the proof of the accuracy of the method of crossed spectra. By crossing the echelon grating with L.G. interferometer and utilising the method of coincidences† between the interference points of these satellites, we have measured the distance between the satellites 0 and 365 accurate to four figures and found it to be 365·0. Relying on the echelon spectrum only, which in the present case was better defined than in the former experiments, I have, after repeated micrometric measurements, found it to be 365·0, which exactly coincides with the number obtained by a more accurate but complicated process. It further lends weight to the reliability of echelon spectroscope in analysing the structure of spectrum lines, although the apparatus is more complex and susceptible to changes in the instrumental constants than a L.G. plate, as is evident from the frequent appearance of ghosts in the spectra produced by it.

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* See Note added to the present Paper (p. 114).

† Nagaoka and Takamine, " Phil. Mag., 27, p. 123, 1914.

§ 14. In our former communication, no attempt was made to arrange the satellites according to the difference in wave-number. If they are arranged according to the numbers of the last column in the Table (p. 98), we find that there is a remarkable symmetry in the distribution of the satellites in different lines. In Fig. 2, the first row represents the general aspect of

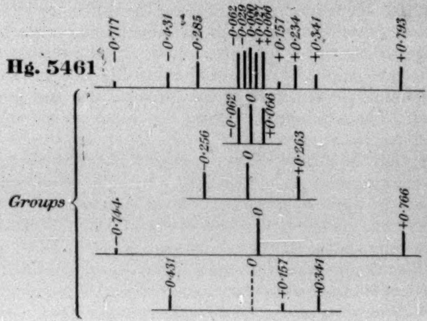


Fig. 2.

the green line, with the intensity taken proportional to the length of the line. Arranging them in groups as indicated in the following four rows, we find a remarkable symmetry in the distribution of the satellites.

The principal line forms a triplet with −0·062 and +0·066 as components; the satellite −0·029 then forms another triplet with the components −0·263 and +0·256 referred to it, being nearly four times the distance between the components of the principal triplet; then +0·027 forms another triplet having the components −0·744 and +0·768 referred to it, the interval being about three times that in the second triplet and 12 times that in the first triplet. The remaining satellites −0·431, +0·157 and +0·341 have apparently no regularity. Further, the components of the triplet have symmetry in the intensity of lines in the first and second groups, but in the third this is no longer the case. The wave-number in the last row is in a singular ratio. The difference in the wave-number intervals counted from the principal of the first positive and negative component $=0{\cdot}274$, and the difference between the first and the second positive components $=0{\cdot}184$; these numbers are in the ratio 3:2.

In addition to this, we find that the interval between the satellites

$$-0{\cdot}717 \text{ and } +0{\cdot}341=1{\cdot}058$$
$$-0{\cdot}285 \text{ and } +0{\cdot}793=1{\cdot}078.$$

The mean of these two numbers $=1{\cdot}068$. It will be afterwards shown that for several mercury lines, the difference in wave-numbers of satellites is constant and about equal to the above number, so that this is not characteristic of the green line only.

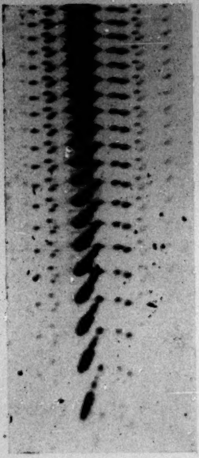
§ 15. The complexity of the violet line $\lambda\,4{,}359$ is well known, but the multiple structure of the principal line was not yet studied. The arrangement being the same as before, the crossed echelon spectrum obtained by short exposure showed at once that the principal line is complex, but the components are so close together, that 10 m.A.U. were difficult to separate. Crossing the plate with the F.P. interferometer, the complex structure was made out as shown in Figs. 3 (A, B). The interference points are arranged in a manner similar to those in the green line, the principal forming a triplet, of which the middle component is rather weak compared with the side components, each of them being associated with a strong satellite. Perhaps the middle of the triplet, though weak, can be taken as the principal line. The lines of the group are distant by the following amount:—

$$-17{\cdot}9 \qquad -5{\cdot}8 \qquad 0 \qquad +5{\cdot}7 \qquad +16{\cdot}2 \qquad \text{m.Å.U.}$$

For determining the position of other satellites, I have used crossed echelon spectra. The table given on next page gives the comparison referred to the strong outermost satellite which is easily oriented. The difference of wave-numbers in the last column are not calculated from what appeared to be the principal line, but the initial point is laid at the middle of two strong lines 157·7 and 163·4, so that they form a strong doublet.

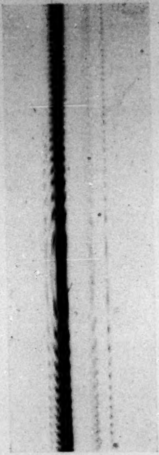
§ 16. In our former experiments with an echelon grating and a L.G. plate, the exposure was limited to 20 or at most 30 minutes, but in the present experiment with an echelon and F.P. instrument, the satellites already found were distinctly visible with exposure of 20 minutes; when it was prolonged to 40 minutes, additional lines appeared, which evidently were not to be identified with ghosts; by exposure for two hours,

Fig. 4A.
λ 4,359.



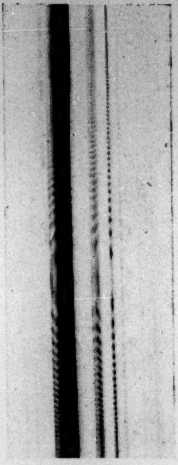
Ech.+L.G.
Exp. 60 min.
(Compare with Fig. 22 of the former communication, Vol. 25 of the "Proceedings.")

Fig. 4B.
λ 4,359.



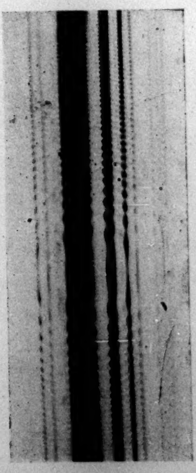
e=5·0 cm.
Ech.+F.P.
Exp. 20 min.

Fig. 4C.
λ 4,359.



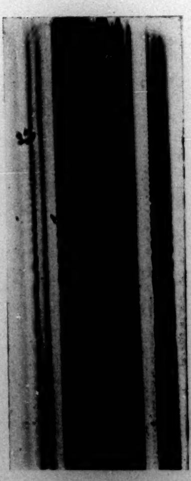
e=10·5 cm.
Ech.+F.P.
Exp. 30 min.

Fig. 4D.
λ 4,359.



e=7·0 cm.
Ech.+F.P.
Exp. 40 min.

Fig. 4E.
λ 4,359.



e=8·0 cm.
Ech.+F.P.
Exp. 120 min.

To face p. 101.

Von Baeyer.	Nagaoka and Takamine.		Nagaoka.	
L.G.+L.G.	Echel.+L.G.		L.G.+ or Echel.+F.P.	
...	...	...	−78	+1·254
...	...	...	−57	+1·082
0	0	0	0	+0·844
52	53	53	53	+0·566
66	67	66	65	+0·502
114	116	115	114	+0·245
142	142	140	139·8	+0·110
...	...	...	151·9	+0·047
159 ?	161 ?	163 ?	157·7	+0·015
...	...	...	163·4	−0·015
178	181	180	179·6	−0·099
187	193	189	190	−0·154
203	206	206	207	−0·244
273	267	268	268	−0·566
...	...	...	318	−0·834
344	344	345	343	−0·956
...	...	...	383	−1·176
...	...	...	398	−1·251

the group about the principal was completely darkened, but new lines became distinctly visible. Trying similar experiment with L.G. plate and echelon, I noticed traces of faint lines apparently near the strong lines of the principal group, and though much obscured by them, the course of the interference points indicated that the faint dots belong to echelon spectra different from the principal. The photograms are shown in Figs. 4 (A, B, C, D, E). These satellites are nearly coincident with those found by Wood with a 6 in. concave grating. Although the identification is not perfect, the difference is probably to be ascribed to the error of observation. These are −78, −57, +318, +383 in the above Table.

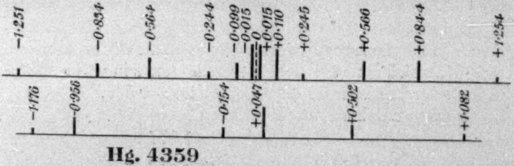


Fig. 5.

§ 17. Plotting the differences of wave-number (Fig. 5), we get two rows of which the upper one shows remarkable symmetry. The intensity of −0·099 is somewhat less than that of +0·110, and for these lines the symmetry is not so exact as for other satellites in the same row. The components near the initial point, *i.e.*, −0·244, −0·099, −0·015 and +0·015, +0·110, +0·245, evidently form a sextet, and the outer components −1·251, −0·834, −0·564 and +0·566, +0·844, +1·254 another sextet, which is more symmetrical than the inner. The wave-number interval of the outer sextet of λ 4,359 is so related that the mean interval of the inner components 0·274 is to the mean of the outer 0·414 in the ratio 2:3. This regularity is indeed remarkable and will no doubt have some intimate connection with the atomic structure. It must not, however, be forgotten that the principal line, which is evidently represented by the triplet −0·015, +0·015 and +0·047 is split into a sextet in magnetic fields, so that the structure here exhibited may have an important relation to the Zeeman effect of the lines. It is also to be noted that the strong satellites −0·099 and +0·110 show the same mode of separation.

§ 18. The lines in the second row have apparently no symmetry, but some of them show striking coincidences in the frequency difference with components of both rows. As to the row of satellites having no apparent symmetry, we can detect the same regularity as for the green line. The difference in absolute wave-number for the first positive and the second negative components =0·909, and the same quantity between the first two positive components =0·455; these numbers are in the ratio 2:1. Further the interval between the principal −0·015 and the second negative component of the same row =1·161, which is nearly double the difference in wave-number of the second and third positive satellites =0·580.

The wave-number interval between

−1·176 and −0·099=1·077
−0·956 ,, +0·110=1·066
−0·834 ,, +0·245=1·079
−0·564 ,, +0·502=1·066
−0·244 ,, +0·844=1·088
+0·015 ,, +1·082=1·067
Mean .... =1·074

A similar interval in λ 5,461 (§ 14) has a mean interval 1·068, which is almost equal to the number found for λ 4,359. Moreover, the interval between

−1·251 and −0·244=1·007
+0·245 ,, +1·254=1·009
Mean ...... =1·008

The difference between these two numbers =0·066, which is the interval between +0·566 and +0·502, and also corresponds to the interval in the triplet found in the green line. These numerical coincidences are apparently not a mere chance, as will be further shown for the line λ 4,047.

§ 19. Experiments similar to those on the lines λ 5,461 and λ 4,359 were made on the violet line λ 4,047. The optical constants of the L.G. plate were such that several satellites came on the same line with the principal, so that even with crossed spectra, the structure was not distinctly observed as with the lines above described. Crossed echelon and F.P. interferometer spectra were more definite, and gave points, which showed that the principal line is a triplet accompanied by a satellite. The wave-number differences counted from the principal are as follows :—

−0·715, −0·362, −0·025, 0, +0·024, +0·075, +0·290, +0·375, +0·708.

It is rather difficult to estimate the resolving power used in the experiment, as the reflection is much enfeebled for such short waves and the absorption in the echelon is much increased, but at the distance of 6 cm. or 7 cm. it will be over $1{\cdot}5\times 10^6$, which is no doubt sufficient to effect the above resolution. Plotting the above number in Fig. 6 we find that there is a triplet at the middle, and two lines on both sides are symmetrically situated about the principal. There are two other lines which are evidently not included in the same group. With this line, the interval in wave-numbers between

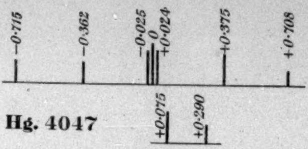


FIG. 6.

−0·715 and +0·375=1·090
−0·362 and +0·708=1·070

Mean ...... =1·080

The occurrence of this number in λ 5,461 and λ 4,359 was already noticed ; moreover, the ratio between the mean components 0·368 and 0·712 to 1·080 is as 1:3 and 2:3 respectively.

§ 20. The similarity in the distribution of the satellites and the occurrence of the same wave-number intervals may be attributable to the character of the second subordinate series to which the three lines here mentioned belong. Arranging the wave-number intervals in another manner, we find that there are many numbers which are common to two of them or to all the three lines. As to the initial position, we shall take +0·793 for λ 5,461, +0·844 for λ 4,359, and +0·708 for λ 4,047. Tabulating the lines we find the following coincidences indicated by dots :—

5,461	4,359	4,047
	+410	
	+238	
0	0	0
	−278	
	−342	−333
−452		−418
−559		
	−599	
−636		−633
		−684
		−708
−727	−734	−733
−766		
−793	−797	
−822	−828	
−855	−859	
	−943	
−1,078	−1,088	−1,070
−1,224		
	−1,403	−1,423
−1,510		
	−1,678	
	−1.800	
	−2,020	
	−2,095	

These numbers are expressed in units 1,000 times smaller than those already used. The table shows that the three lines are not independent of each other, and the satellites are not arbitrarily distributed. It is a question why such coincidences are not found with reference to the principal lines.

Owing to the symmetry in the distribution of the satellites in λ 4,359, the coincidence above stated can be found in the same manner by taking the initial point at −1,678 in the above schedule. Most of the numbers are reproduced, so that by proceeding backwards, we meet with most of the numbers found for λ 5,461 and λ 4,047 already given.

§ 21. Turning our attention to the lines of the other series, we see that in the mercury line λ 5,769 (Fig. 7) there is a triplet about the principal with the wave-number difference −0·139 and +0·147. It is also the case with the line λ 4,078 (Fig. 8), in which the difference amounts to −0·283 and +0·285; this interval is nearly double that between the outer components in the triplet of λ 5,769. We notice that the interval between the two outer negative components of λ 4,078 is 0·146, which is equal to the distance of the first positive component from the principal in λ 5,769. Moreover, the second positive component of the line is distant from the principal by 0·360, which is nearly double the distance of the principal line λ 4,078 from the first negative component.

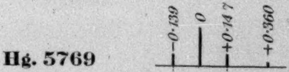


Fig. 7.

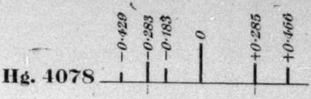


Fig. 8.

Further, we notice that the intervals between the three lines −0·137, +0·147 and +0·360 in λ 5,769 being 0·284 and 0·213 are in the ratio 4:3. Obviously the components of λ 4,078, −0·183, 0, +0·285, +0·466 (=0·285 +0·181) form a quartet. Further experiments on the principal line of 4,078 showed that it consists of three other satellites on the side of lower frequency, at a constant interval of about 0·039, but as these components were difficult to observe with the L.G. plate and F.P. interferometer combination, the position is not indicated in the figure. The structure of the principal line λ 5,769 was difficult to analyse on account of the low sensitivity of photographic plate; some photograms showed that it is a close triplet.

§ 22. The structure of the mercury lines λ 3,663, 3,655, 3,650 belonging to the first subordinate series is shown in Fig. 9, and is taken from the measurements of Mr. Takamine.* Here we notice the same constant difference in the wave-number.

For 3,663, interval between −0·678 and +0·395=1·073
„ 3,655 „ „ −0·546 „ +0·518=1·064
„ 3,650 „ „ −0·374 „ +0·712=1·086

Mean =1·074

This interval was already noticed in connection with the lines belonging to the second subordinate series, and seems to be characteristic of most mercury lines. In the above-mentioned lines there is another interval which is common to them; the difference in wave number

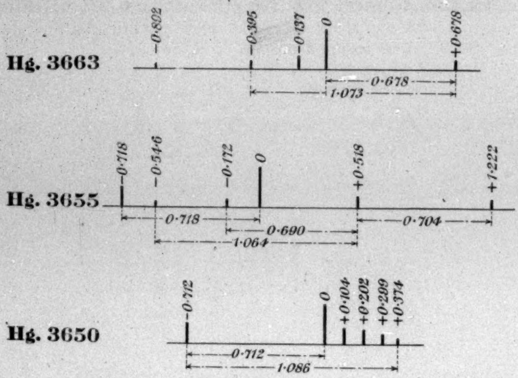

−0·802
−0·395
−0·137
0
+0·678
Hg. 3663
0·678
1·073
−0·718
−0·546
−0·172
0
+0·518
+1·222
Hg. 3655
0·718
0·690
0·704
1·064
−0·712
0
+0·104
+0·202
+0·299
+0·374
Hg. 3650
0·712
1·086

Fig. 9.

For 3,663, between −0·678 and 0=0·678
„ 3,655 „ −0·718 „ 0=0·718
„ 3,655 „ −0·172 „ +0·518=0·690
„ 3,655 „ +0·518 „ +1·222=0·704
„ 3,655 „ 0 „ +0·712=0·712

Mean =0·700

The ratio of this number to another constant 1·07 is about 2:3. We meet with a similar number in λ 4,047, in which the outermost components have the mean value 0·711 which stands to 1·07 in the ratio 2 : 3 more accurately than for the three lines just discussed.

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* Takamine, "Proc.," Tokyo Math. Phys. Soc., 8, p. 309, 1915.

It is worth noticing that the intervals given by $l\frac{c}{3}$, $m\frac{c}{4}$, and $n\frac{c}{5}$, where $c=0{\cdot}71$ and l, m, n are whole numbers, are found scattered among the satellites of the eight mercury lines above mentioned, for $l=1$ to 4, $m=1$ to 7, and $n=1$ to 7.

The second positive satellite of the line 3,663 is nearly three times as distant from the principal as from the first, and the negative satellite is five times the same amount. The first positive satellite of the line 3,655 is distant from the principal by three times as much as the first negative, and the second negative satellite appears as the difference between the first and the third satellite, thus forming a quartet. The difference between the first and the second satellite on the positive side is 0·704 and is nearly equal to the distance of the first positive satellite in 3,650. The components −0·172, +0·518, +1·222 form a triplet.

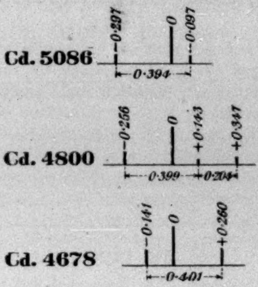

FIG. 10.

§ 23. According to the investigation of Mr. Takamine,* the structure of the cadmium line is simpler, and the regularity more apparent. The lines λ 5,086, λ 4,800, λ 4,678 belong to a triplet series, corresponding to the mercury lines first discussed. These lines have satellites as shown in the difference of wave-number in Fig. 10.

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* Takamine, "Proc." Tokyo Math. Phys. Soc., 8, p. 51, 1915.

As already noticed, we meet with a constant interval in the wave-number of the satellites as follows :—

For 5,086, interval between −0·297 and +0·097=0·394
„ 4,800 „ „ −0·256 „ +0·143=0·399
„ 4,678 „ „ −0·144 „ +0·260=0·401

Mean =0·398

This number evidently corresponds to 1·07 and 0·71 already found for mercury lines. Further, we have to remark that for λ 5,086, the interval between −0·297 and 0 is equal to 3×0·099, while the interval between the principal and the positive satellite is 0·097 ; similarly for λ 4,800 the interval between the negative and the first positive component =0·399 is nearly double the interval 0·204 between the components. With λ 4,678, the distribution of the satellites is reverse to that for 4,800, for in place of −141 and +260 in the former, we find +143 and −256 in the latter. Thus the satellites of these three lines seem to be mutually dependent, as already noticed for the mercury lines This singular fact would have been noticed by Mr. Takamine, if he had reduced his measurements to wave-numbers. It is customary to give the position of the satellites in wave-lengths, but from the theoretical point of view it is advisable to give the wave-number.

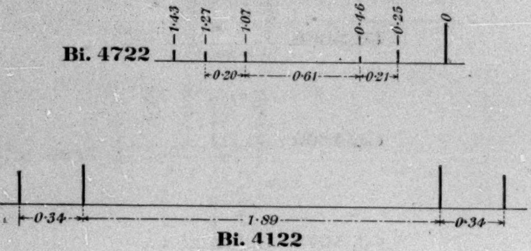

Fig. 11.

§ 24. Similar features are shown by the bismuth lines λ 4,722 and λ 4,122. They are shown in Fig. 11 : the interval between the components −1·27 and −1·07 =0·20 and that between −0·46 and −0·25 =0·21, while that between −1·07 and −0·46 =0·61, being three times the above amount. λ 4,122 consists of a strong doublet at an interval of 1·89, having the satellites at equal intervals of +0·34 and −0·34 from each of them; it is also to be noticed that the interval between the doublets in the latter line is nearly three times that of the former.

§ 25. It would be premature to inquire into the cause of the regularity of the satellites mentioned in the preceding pages, as the data on which the discussion rests are as yet scanty, and the effects of the electric and magnetic fields on the behaviour of spectrum lines are not yet well known. The presence of magnetic and electric fields in atoms is a matter beyond doubt, and changes in interatomic fields will be felt in the vibrations of electrons. It is only recently that Stark has been able to establish definitely the effect of external electric field on emission lines, while that of the magnetic field, although well known for about 20 years, has not yet been satisfactorily explained, excepting that of the normal triplet, which forms only one of the numerous types of separation. If the said effect had been more accurately investigated, it would perhaps not be difficult to found a theory of the existence of the satellites, as the most plausible explanation seems to lie in the change of the quasi-elastic force giving rise to light vibrations, due to the variation of the interatomic fields. The theory of the effect of an electric field on the spectrum line has been discussed by Voigt,* and of the joint action of the electric and magnetic forces by Zeeman.† The investigation of positive rays by Sir J. J. Thomson has revealed the existence of atoms of mercury, which have lost one and sometimes eight electrons at one time. The change in the electric field of mercury atoms by the liberation of so many electrons will have sufficient influence on the radiation by producing an appreciable disturbance of the field, that the wave-length is slightly altered; most probably the change thus caused in the electronic configuration will alter the atomic magnetic field to such a degree, that it will sensibly affect the radiation sent out by the mercury atoms. If the investigation begun by Thomson be extended to different elements, it will shed much light on the nature of the spectrum lines, but in the present stage of our knowledge it is rather difficult to advance a new hypothesis as regards the presence of satellites on the basis of the change of atomic electric and magnetic fields.

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* Voigt, "Ann. der Phys.," 4, p. 197, 1901; "Magneto- u. Elektro-optik," 1908.

† Zeeman, "Akad. v. Wetens.," Amsterdam, May 27, 1911.

§ 26. In his theory of the vibrations of electrons in an isotropic quasi-elastic electric field, Voigt has shown, under the assumption of the existence of potential, which contains terms of higher order than that usually assumed in elasticity, that the change of frequency is proportional to the square of the field; observed transverse to the field, the original line is separated into two components of the same frequency, and parallel to the field there is a displacement three times as large as that of the line vibrating perpendicular to the field. Extending the analogy of the Zeeman effect in an electric field, Voigt has further shown that we shall have a quartet. In the application to mercury atoms it is natural to assume that the result will conform to the case in which the electrons are coupled, because the green line is divided into a nonet in a magnetic field, the violet line λ4,359 into a sextet, and λ4,047 into a triplet of double the normal separation, all of which can be explained from the standpoint of the coupled electrons. If we now assume that such coupling takes place in the electric field in the same way as for the magnetic, we can without much difficulty explain the regularity observed in some lines of mercury, if not the whole of it. In other cases, Zeeman's calculation, based on a simple Lorentz's hypothesis as regards the action of magnetic force, lends support to the explanation of the appearance of the satellites in cadmium and bismuth lines.

§ 27. The discussion of the effect of an electric or magnetic field indicates the nature of the polarisation to be observed if the direction of the field be fixed, but in case the seat of such fields lie in the radiating atoms, the polarisation observed becomes promiscuous and cannot be discriminated. With the green line (Fig. 2), taking −0·06 as the initial point, we notice that the zeros of the lines as arranged in the third and fourth row in the figure are distant from it by 0·03 and 0·09, being in the ratio 1:3 as expected on Voigt's theory. Starting from these lines, we have the lines ± 0·26 in the third row and ± 0·75 in the fourth, thus affording a solution in the light of coupled electrons. The last row consists of an unsymmetrical triplet, which can be caused by the joint action of the electric and magnetic forces as indicated by Zeeman. Another suggestion that we can make of these three symmetrical triplet rows in the diagram is that they are all due to the Zeeman effect, the magnetic field producing these separations being caused by the disturbance in the configuration of electrons in the mercury atom on the expulsion of electrons and their subsequent return to it. But it seems to me that the former way of seeking the solution of the problem is more plausible than the latter; for there are different instances which favour the view that the satellites are due mostly to electric fields. The feeble satellites −0·137 and −0·395 in λ 3,663, +0·172 and +0·546 in λ 3,655, and +0·104 and +0·299 in λ 3,650 are all in the ratio 1:3 and conform to Voigt's simple theory. This is, however, only one aspect of the phenomena; as, for the line λ 3,655, we may consider the group of lines −0·718, −0·546, −0·172, and 0 as forming a quartet having a constant interval of 0·172 at both ends; moreover, the lines −0·172, +0·518 and +1·222 may be looked upon as a triplet with a mean interval of 0·697; as to the line λ 3,650, the said lines may form a group of gradually evanescent lines; thus various arrangements of lines may be considered as possible, for which different solutions can be suggested. If the said lines in λ 3,655 be considered as arranged in a quartet, then they may be due to the action of the electric and magnetic forces which are at right angles to each other. The satellites belonging to the bismuth lines λ 4,722 and λ 4,122 belong to the same type, as indicated in Fig. 11.

As to the line λ 4,047 of mercury (Fig. 6), the middle line may represent the position in the undisturbed state, while the other four lines may correspond to the state in which the coupled electrons vibrate in the electric field, or may be subject to the action of electric and magnetic forces. Near by the principal are two lines +0·024 and +0·075 in the ratio 1:3; they thus seem to belong to the lines conformable to Voigt's theory, but the four lines in the upper row are symmetrical about the principal and not about these two; whence it seems plausible that they are produced in the uncoupled condition. The presence of two symmetrical sextets in λ 4,359 (Fig. 5) is not easy to explain; on Zeeman's theory the presence of a dissymmetrical triplet is made possible, and when the presence of the line in zero field is considered as possible, there results a symmetrical quartet when the magnetic force is perpendicular to the electric. If the theory of the Zeeman effect be perfected, so that it can explain different modes of separation, such as a quartet, sextet or nonet, we shall be in a position to explain the distribution of the satellites in λ 4,359. If we exclude the outermost components, which are very faint, and not observed by most experimenters, we can consider the lines −0·834 −0·564 and +0·566, +0·844 as forming a symmetrical quartet, and −0·244, −0·099, and +0·110, +0·245 as forming another quartet; these may be supposed as due to the vibrations of coupled electrons in two different electric fields, which may arise during the liberation of electrons from the radiating atom or return to it. Similarly, if the outer satellites in the second row be not taken into account, the lines form a dissymmetrical triplet and may be due to the electric and magnetic forces. Good instances of a dissymmetrical triplet are found in the cadmium lines λ 5,086 and λ 4,678, while the mercury line λ 5,769 and the cadmium line λ 4,800 form dissymmetrical quartets; singular enough is the fact that in the former the intervals among the satellites are in the ratio 3:4 and in the latter 1:2; this makes us consider the cause of these lines as due to some action or arrangement of electrons, which give rise to the above mentioned regularity.

I cannot as yet lay much importance on the theories above given; they may be looked upon as having some significance in the theoretical aspect of the problem.

§ 28. When we consider the different causes which introduce the regularities above mentioned, we naturally fall back on the supposition that the primary cause is of electromagnetic origin. As stated before, the researches of Thomson seem to give some hints to the origin of such forces. As regards the mercury atom we know that it gives out a single electron or eight electrons at once; the former is the usual mode of ionisation, and the emission of the principal lines will also be the consequence of the expulsion of the electrons giving rise to electric and magnetic disturbances. As to the latter way of ionisation, it is peculiar to mercury, and the disturbance arising from it will be of such an amount that the field in which the atomic constituents are placed will be subject to changes, by which the electrons emit intense electromagnetic waves. How often such expulsions of electrons take place compared with the number of expulsions of a single electron is not known, but most probably it will be less frequent, and though the emission may be intense, the total effect will not be so great as for the ordinary single electron emission. Owing to the violent change consequent upon the expulsion of several electrons, the electric field of the atom will be influenced by it, and the vibration of electrons modified to such a degree as to give rise to satellites. It will be at the moment of disruption that the electrons are subject to the most violent acceleration, and are in a state of giving out the most intense radiation. It therefore seems to me probable that the principal lines are due to the consequence of the expulsion of an electron from the atom, and the satellites are due to the disruption of a ring of electrons. It may, however, be argued that on the gradual return of the electrons to the atom, there will be a new adjustment of the electron configuration in the atom, which calls forth radiation; this will inevitably be the case, but the disturbance caused by the return of the electrons to the atom will generally be small as compared with the expulsion of eight at one time, so that the result is not so great as to affect the photographic plate.

§ 29. The frequent occurrence of the common difference in the wave-number for lines of different wave-length calls for a new explanation. According to Bohr, the radiation emitted during a transition between two stationary states of an electron system in an atom is determined by

$$h\nu = A_1 - A_2,$$

where h is Planck's constant, ν the frequency, and A_1 and A_2 are the energies of the system in two stationary states. For a satellite of frequency $\nu + \delta\nu$, a similar equation must hold, so that we shall have

$$h\delta\nu = \delta A.$$

The above fact shows that for different values of ν, $\delta\nu$ has sometimes constant values, and consequently δA is common to these satellites. In other words, the energy difference between two stationary states which give out radiation corresponding to the satellites is constant. Assuming that the change in the stationary states of the emitting system is the effect of electric and magnetic fields, the energy possessed by the system in these states will be altered in such a way that the amount is still some multiple of a certain quantum, which is different from h.

On this hypothesis, the occurrence of such parts in the difference between the wave-numbers of satellites is a natural consequence. This is well exemplified in most of the lines discussed in the preceding pages; the numbers 1·07 and 0·70 found in the mercury lines, and 0·40 in the cadmium lines are remarkable instances of the existence of the quantity just discussed.

Judging from the fact that such common differences are characteristic of the lines belonging to the same series, it would appear that the lines belonging to the same series emit multiples of quanta, which are peculiar to them. If we suppose that the electrons are arranged in rings about a central nucleus, and that they revolve in circular orbits, the above conclusion indicates that the stationary orbits do not vary continuously, but by jumps. This is necessarily due to the quantum change in energy, which will be emitted in the case of mercury atom, when an electron is liberated from it, or when eight electrons are expelled, or on subsequent return of electrons till the atom becomes ultimately electrically neutral. These considerations lead to the conclusion that the existence of numerous satellites in mercury lines is not at all singular, but is a natural consequence of the mode of ionisation peculiar to the element. If the experiment on the positive rays be extended to various elements and its consequence examined in the light of the theory here sketched, we shall perhaps be able to decide the question as regards the origin of the satellites. To account fully for the existence of the satellites and the regularity of their distribution, we are still in want of sufficiently accurate data as to the position of the satellites in order to test the consequences of the hypothesis which has been here advanced.

Note on the Structure of the Green Line of Mercury and the Terms of Correction in Using a Concave Grating.

The structure of the green line of mercury as revealed by Michelson's grating* ($9\frac{3}{8}$ in. × $2\frac{7}{8}$ in.) of 20 ft. focal length in the sixth order spectrum presents some points of discrepancy between that obtained by means of interferometers, echelon gratings, Lummer-Gehrcke plate, or Fabry-Perot apparatus. The principal difference is the obliteration of the strong satellites $-19{\cdot}9$ and $+18{\cdot}4$m.Å.U. from the principal line, which are fused into one line, although the resolving power of the grating would have been sufficient to separate them; the triplet structure of the principal would, however, have been difficult to resolve. This, I believe, is due to terms of correction, which, for gratings of such size account for the overlapping of the neighbouring lines.

Take the x-axis in the direction of the normal to the grating at its middle point, and y, z axes in the tangent plane at the point of the sphere, on which the lines parallel to the z-axis are ruled. Referred to these axes, let the co-ordinates of the source of light A be a, b, o; of the point A' at which a spectrum of hth order is formed a', b', o; and of a point P on the grating x, y, z. Then $AP^2=(a^2+b^2)-2\,by\,-2ax+x^2+y^2+z^2$;

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* Michelson, "Proc." American Phil. Soc., 54, p. 137, 1915.

putting $r^2=a^2+b^2$, and introducing the condition that P lies on a sphere of radius ρ, we get

$$AP^2=r^2-2by+\left(1-\frac{a}{\rho}\right)x^2+\left(1-\frac{a}{\rho}\right)y^2+\left(1-\frac{a}{\rho}\right)z^2,$$

and $A'P^2=r'^2-2b'y+\left(1-\frac{a}{\rho}\right)x^2+\left(1-\frac{a'}{\rho}\right)y^2+\left(1-\frac{a'}{\rho}\right)z^2.$

In the present problem, y may be considered as a small quantity of the first order, and x as cf the second order, while in practice z is intermediate between x and y. Expanding in powers of y, we obtain

$$AP=r-\frac{b}{r}y-\frac{a}{2r}\left(\frac{a}{r^2}-\frac{1}{\rho}\right)y^2-\frac{ab}{r^2}\left(\frac{a}{r^2}-\frac{1}{\rho}\right)y^3+\ .\ .\ .\ .$$

with similar expression for $A'P$. If we make the terms containing y^2 and y^3 evanescent in these two expressions, we arrive at the usual Rowland's arrangement $r^2=a\rho$ and $r'^2=a'\rho$, by which the points A and A' lie on a circle of half the radius of the sphere. When this condition is fulfilled,

$$1-\frac{a}{\rho}=1-\frac{a^2}{r^2}=\frac{b^2}{r^2},$$

and $$AP^2=r^2\left\{\left(1-\frac{b}{r}y\right)^2+\frac{b^2}{r^4}(x^2+z^2)\right\}.$$

which, on expansion, gives

$$AP=r-\frac{b}{r}\left(1-\frac{b^2}{2r^4}z^2\right)y+\frac{1}{2}\frac{b^2}{r^3}(x^2+z^2).$$

Putting the approximate value $x=\frac{ay^2}{2r^2}$, and denoting the angle of incidence by θ, and the angle of diffraction by θ',

$$AP=r-\sin\theta\{1-\frac{\tan^2\theta}{2}\left(\frac{z}{\rho}\right)^2\}y+\frac{\tan\theta\sin\theta}{2}\ \frac{z^2}{\rho}+\frac{1}{8}\tan\theta\ \sin\theta\frac{y^4}{\rho^3}$$

$+\ .\ .\ .$ with similar expression for PA'.

Confining our attention to a constant value of z, we get the equation of finite difference with respect to y.

$$\Delta(AP+PA')=\{-(\sin\theta+\sin\theta')+\left(\frac{\tan^2\theta\sin\theta}{2}+\frac{\tan^2\theta'\sin\theta'}{2}\right)\left(\frac{z}{\rho}\right)^3$$

$$+\left(\frac{\tan\theta\sin\theta}{2}+\frac{\tan\theta'\sin\theta'}{2}\right)\left(\frac{y}{\rho}\right)^3\}\ \Delta y.$$

Let the successive intervals of the ruling along the chord be $\varepsilon = \Delta y$; then, for a spectrum of h's order, we shall have

$$\varepsilon\left\{(\sin\theta+\sin\theta')-\left(\frac{\tan^2\theta\,\sin\theta}{2}+\frac{\tan^2\theta'\,\sin\theta'}{2}\right)\left(\frac{z}{\rho}\right)^2 - \left(\frac{\tan\theta\,\sin\theta}{2}+\frac{\tan\theta'\,\sin\theta'}{2}\right)\left(\frac{z}{\rho}\right)^3\right\}=\pm h\lambda.$$

According to the usual way of treating a grating spectrum, the corrections in the second and third terms are not taken into account, so long as they do not exceed $\pm\frac{\lambda}{4}$. If the line be not associated with satellites the assumption would be admissible, but in case the structure of the line is complex, the effect of these terms is to introduce admixture of different lines, and the feeble satellites are liable to be effaced; with strong satellites, the demarcation between the lines becomes indistinct, so that they are amalgamated into a wide line with vague boundaries.

Let λ_0 be the wave-length for $y=0$ $z=0$; then corresponding to the same angles, and for y and z, $\lambda=\lambda_0+\delta\lambda$, where

$$\delta\lambda=\pm\frac{\varepsilon}{2h}\left\{(\tan^2\theta\,\sin\theta+\tan^2\theta'\,\sin\theta')\left(\frac{z}{\rho}\right)^2 + (\tan\theta\,\sin\theta+\tan\theta'\,\sin\theta')\left(\frac{y}{\rho}\right)^3\right\}.$$

The deviation from λ_0 increases very rapidly, as the portion of the grating at some distance from the middle is utilised, especially when the angle θ increases and the ruled space amounts to a few hundredth part of the focal length of the grating. The following tables, giving the corrections, at once show the importance of taking these terms into account.

Table of $\frac{\sin\theta\,\tan\theta}{2}\left(\frac{z}{\rho}\right)^2\times 10^7$.

θ $\frac{z}{\rho}=$	1/500	1/400	1/300	1/250	1/200	1/150
10°	0	0	0	0	1	1
20°	1	1	3	4	6	10
30°	3	5	9	13	21	37
40°	9	14	25	36	57	101
50°	22	34	60	81	136	241
60°	52	81	144	208	324	577
70°	142	221	394	567	885	1,576
80°	634	991	1,760	2,534	3,977	7,039

Table of $\frac{\sin\theta\tan\theta}{2}\left(\frac{y}{\rho}\right)^3\times 10^7$.

θ $\frac{y}{\rho}=$	0·01	0·02	0·03	0·04	0·05
10°	0	1	4	10	19
20°	1	5	17	40	78
30°	1	12	39	92	180
40°	3	22	73	173	336
50°	5	37	123	292	570
60°	8	60	202	480	937
70°	13	103	348	830	1,615
80°	28	223	754	1,786	3,491

These tables show at once how the corrections arising from large values of y and z increase very steeply as the angle of incidence is increased. In addition to this, it must not be omitted that the presence of the term $\left(\frac{\tan\theta\sin\theta}{2}+\frac{\tan\theta'\sin\theta'}{2}\right)\frac{z^2}{\rho}$ in the expression for $AP+PA'$ gives rise to the further intermixture of lines, if in practice we gather the light coming from different points of the ruled spacing.

An example may be cited for a grating of 20 in. focal length with 11,700 lines to the inch; for the green line of mercury λ 5,461 in the sixth order spectrum, a simple calculation shows that $\theta=\theta'=48{\cdot}2^\circ$, and for $y=5$ in., $\delta\lambda=24$m.Å.U., which is greater than the distance of the satellites lying close to the principal line; for $y=10$ in., $\delta\lambda=213$m.Å.U., which greatly surpasses the limit of error usually admitted in wave-length measurements.

It is therefore not at all surprising to find that in the photogram of the green line of mercury obtained with Michelson's 10 in. grating, the principal line is amalgamated with its neighbouring satellites into a single broad line, which far surpasses the other lines in intensity. The ambiguity in the position of the satellites arising from the overlapping of spectra of different orders usually observed with interferometers of high resolving power, and which can be discriminated by the method of crossed spectra, is entirely got rid of in spectra obtained with a concave grating. This is a great merit of the grating with large ruled spacing, and physicists are under great obligations to Michelson that he has brought his instrument to such perfection that no ghost can be detected, and the ruled space is so large that most of the satellites can be determined with accuracy and without ambiguity. What I want to indicate in this note is that there are terms of correction, which make the resolving power of the instrument less than that calculated by the usual formula, and that large values of the angle of incidence and of diffraction in such a grating lead to considerable deviations.

These considerations lead to the conclusion that for the exact delineation of fine structure of lines, a concave grating with a large ruled spacing is not so effective as interferometers of high resolving power, due to the gradual intermixture of neighbouring lines. To remedy this defect, the only course that may be followed is to increase the radius of the sphere on which the lines are ruled. Another means would be to rule at smaller intervals, and thus have such a large number of lines within a narrow space as would make the terms of correction above discussed less disturbing in the result.

ABSTRACT.

The Paper describes a further development of the work done by the author and Mr. Takamine on the distribution of the satellites of the mercury lines. The apparatus employed was a Fabry-Perot interferometer crossed with an echelon or Lummer-Gehrcke plate. An account of the theory governing the interpretation of crossed spectra is given.

It is shown that much of the discord between the results of various observers of these satellites is due to the unsatisfactory nature of the principal line as a datum from which to define their positions, and that if the distances be measured from one of the distinct satellites, good agreement is obtained.

If these separations be expressed as differences of wave-number, instead of wave-length, a remarkable symmetry in their distribution becomes apparent. For example, among the satellites of the green line, 5,461, can be found three groups of symmetrical triplets, of which the wave-number differences are in the simple ratios 1 : 3 : 12.

Similar results are obtained for other lines, the principal component of λ4,359 being shown for the first time to consist of a triplet, of which the middle component is relatively weak.

A similarity in the distribution of the satellites exists for all the lines examined, and certain wave-number intervals are common to all.

It is observed that the intervals given by $l\frac{c}{3}$, $m\frac{c}{4}$, and $n\frac{c}{5}$, where $c=0{\cdot}71$, and l, m, and n are whole numbers, are found among the satellites of the eight lines quoted for $l=1$ to 4, $m=1$ to 7, and $n=1$ to 7.

The possible origin of satellites is discussed in the light of the Zeeman and Stark effects, the magnetic or electric fields being electronic in origin, and many of the results are shown to be consistent with Voigt's theory of the vibrations of electrons in an isotropic, quasi elastic, electric field. The suggested explanations can only be tentative as yet, pending further development of our knowledge of the theory

of the Zeeman effect, and the extension of Sir J. J. Thomson's work on positive rays to a larger number of substances.

The second part of the Paper consists of a mathematical investigation of terms of correction in the theory of large sized concave gratings, which prevent the theoretical resolving power from being realised in practice.

DISCUSSION.

Dr. H. S. **Allen** thought the Paper contained many interesting and important points. That simple numerical relationships had been shown to hold for the separations of the satellites was likely to prove very helpful to those who were trying to solve the constitution of the atom. He was interested in the discussion of the part which might be played by the magnetic fields of atomic origin in these phenomena. He had himself held this idea for some time.

Dr. R. S. **Willows** did not quite understand the statement made at the beginning of the Paper that there was, as yet, no observations of the effect of an electric field on the satellites, in view of the experiments of Stark, Lo Surdo and others.

Prof. **Nicholson**, in reply, said he presumed the statement was intended to refer to the mercury satellites only.

VII. *A Clock of Precision.* *By* C. O. BARTRUM, *B.Sc.*

RECEIVED DECEMBER 6, 1916.

I. INTRODUCTION.

II. DESCRIPTION.

1. The Master Pendulum.
2. Impulse Pallet.
3. Gravity Impulse Lever.
4. Releasing Electromagnet. Circuit I.
5. Restoring Lever.
6. Restoring Electromagnet. Circuit II.
7. The Slave Clock.
8. The Counting Mechanism.
9. The Rating Spring.
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11. The Synchronising Mechanism.
12. The Principle.
13. False Synchronism overcome by negative backlash.
14. Description of Synchronising Mechanism.
15. Mode of working.

III. MATHEMATICAL DISCUSSION.

Table of Symbols.

16. The Figure of the Impulse Pallet. Energy conveyed with each Minute Impulse.
17. The Energy Required to Maintain the Master Pendulum.
18. The Couple Retarding the Pendulum due to Friction of Impulse Wheel.
19. The Rating Spring.
20. The Method of Synchronising.
21. The Timing Wheel.

IV. DISCUSSION OF CERTAIN CAUSES OF VARIATION AND THEIR PROBABLE EFFECTS ON THE RATE.

22. Possible Changes of Condition.
23. Circular Error.
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25. Application to Wheel Friction.
26. Application to Impulse.
27. Change of Value of Friction of Impulse Wheel.
28. Change of Instant of Release of Impulse Lever.
29. Shift of the Suspension Sideways.
30. Change of Conditions of Impulse Lever.
31. Change of Effect on the Pendulum of the Fall of the Wheel.
32. The Best Amplitude.

V. CONCLUSION.

I. INTRODUCTION.

In designing a pendulum clock provision has to be made for automatically counting the swings and for maintaining the arc of the pendulum. In general practice these are effected by some form of release, actuated by the pendulum itself, or of contact with the pendulum whereby the vibrations are recorded and the signal given to the motive mechanism that the pendulum is in the position to receive its impulse. Such methods are unobjectionable so far as the disturbance thereby caused to the free swing of the pendulum is constant. In practice this is impossible.

In the clock described below, with a view to this mechanical disturbance, and the variations of this disturbance, being reduced to a minimum, a division of labour is introduced, an auxiliary or slave clock recording the time and releasing the impulse while a master pendulum controls the slave clock.

The master pendulum, rigidly suspended, is intended to be surrounded with an air-tight case and placed below ground or in a locality as far as possible free from vibration and from change of temperature. The slave clock, connected with the master pendulum with electric wires, can be placed in the laboratory or observatory with only ordinary precautions against disturbing vibrations and variations of temperature.

When the writer designed this clock he was unaware that the method of relieving the regulating pendulum by throwing some of its labours upon a slave clock had been previously applied. He has since learned that the clock built for the Royal Observatory, Cape Town, during the régime of Sir David Gill was on this principle. This latter clock, however, was not a practical success, owing to the uncertainty of a delicate electric contact, and was discarded. No description of the Cape clock has been published, but the account given the writer by Sir David Gill showed it to differ in its application of the slave clock method from the clock now being described.

II. DESCRIPTION.

1. *The Master Pendulum.*—The controlling pendulum vibrates seconds, and is compensated for temperature, the compensation being adjusted for conditions where the surrounding air does not change in density with temperature.*

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* The master pendulum in the model is by Riefler, of Munich, with nickel steel rod of low coefficient of expansion and brass bob. No details are discussed of the method of maintaining the constancy of air-density in the pendulum case, as they are not essential to the subject of the Paper.

2. *Impulse Pallet.*—At its lower end is attached an impulse pallet, preferably of stone,* A (Fig. 1), the upper surface of which is horizontal, passing at one end to a slope as shown in the diagram. The figure of the pallet in vertical section is described in § 16.

3. *Gravity Impulse Lever.*†—A gravity impulse lever BC (Fig. 1), pivoted at C, having a small wheel, D, mounted with jewelled pivots at its distal end, is held in a nearly horizontal position by a detent, E. The lever is adjusted by the screw F, so that the periphery of the wheel is as near as possible to the surface of the pallet without danger of its touching.

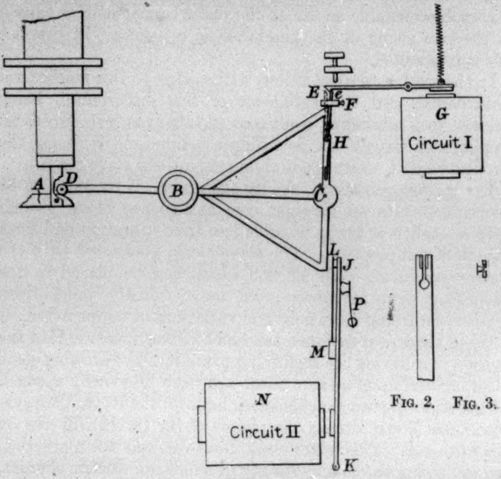

Fig. 2. Fig. 3.

Fig. 1.

The Lower End of the Master Pendulum and its Impulse Mechanism.

4. *Releasing Electromagnet, Circuit I.*—Once in every minute, when the pendulum is approaching its zero position

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* The impulse pallet in the model is of steel.

† The general form of the gravity lever adopted was probably suggested to the writer's mind by the detached gravity escapement of Sir H. H. Cunynghame.

towards the left, an electric current, I, from the slave clock, energising the electromagnet G, releases the impulse lever, the wheel drops through a fraction of a millimetre on to the pallet, rolls down the impulse slope and falls free, giving the pendulum sufficient energy to maintain its swing at the required amplitude.* The period during which the impulse lever is engaged with the pallet is a small fraction of a second,† and for the remainder of the minute the pendulum is free. The catch mechanism E is so arranged that the lever is free from the moment of its release, the pawl, *e*, being driven out of contact with the detent by the spring H. The only forces acting on the impulse lever during its fall are gravity and the reactions at C and D.

5. *Restoring Lever.*—The impulse lever after leaving the pallet falls freely through a short distance, and then strikes a restoring lever, J K. The latter carries on the face that is struck a light spring, L M, electrically insulated at M from the body of the lever, and bearing at its end, L, a wedge-shaped platinum contact piece, L. The restoring lever is split at its end, J (*see* Figs. 2 and 3), the edges of the split being faced with platinum.

6. *Restoring Electromagnet, Circuit II.*—The impulse lever on falling upon the restoring lever drives the platinum surfaces into good contact, thereby closing an electric circuit, II, which, through the medium of the electromagnet N, throws the impulse lever up into its resting position, where it is retained by the detent.

The stop against which the restoring lever rests consists of a spring, P, of such elasticity that the period of electric contact is sufficient for the electromagnet to become saturated before the lever is thrown back. This insures an absence of "chattering" and premature break of the current.

7. *The Slave Clock.*—The slave clock is an ordinary seconds regulator. It is maintained in synchrony with the master pendulum by a method described in (§§ 11 to 15).

8. *The Counting Mechanism.*‡—As mentioned above (§ 4), at intervals of one minute an electric current controlled by this clock releases the impulse lever of the master pendulum. This

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* The semi-amplitude adopted in the model is 1 deg. (*see* § 32).

† In the case of the model this period is about 0·18 second.

‡ The writer owes the idea of a counting wheel to Mr. Hope Jones.

is effected as follows: In the Fig. 4, A is a counting wheel of 30 teeth, B D is a short pendulum with a gathering pawl, C, at its lower end. By means of a horizontal connecting rod, D E, the pendulum moves the wheel one tooth every complete vibration. On the face of the wheel is a platinum pin, F, which at one position of the wheel passes between two platinum contact points, G, closing the electric circuit I. The length of the connecting rod is so adjusted that this momentary closing of the electric circuit is effected when the slave pendulum is at its zero position. The length of the short pendulum B D bears such a relation to the radius of the wheel, and the normal amplitude of the point of the main pendulum which drives it, that the gathering pawl is free of the wheel except near the middle of its path. A light lever, H J, ending in a thin flat spring, J, presses on the teeth of the wheel and also functions as a ratchet pawl. The pressure of this spring can be adjusted by altering the tension of the spiral spring K. It is important that this counting mechanism be as light and as little absor-

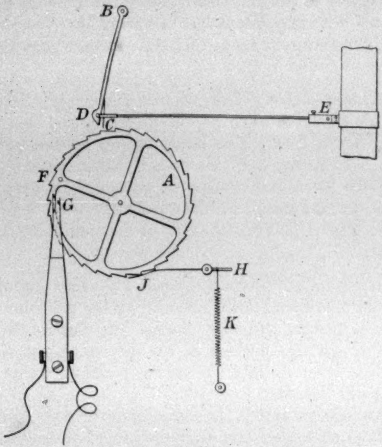

Fig. 4.—The Counting Mechanism.

bent of energy as possible, in order not to overburden the pendulum and the train of the clock.*

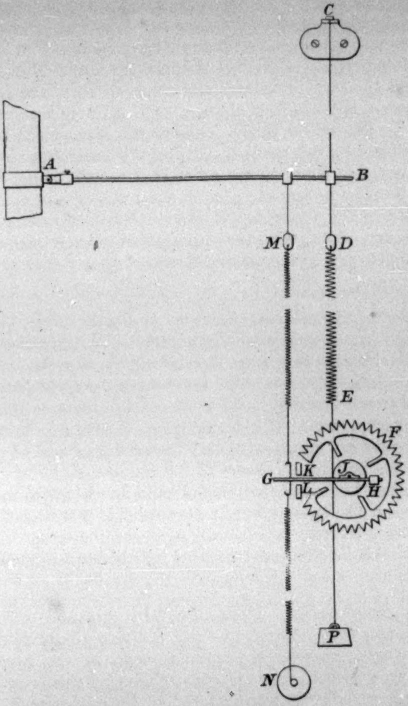

Fig. 5.—The Principal and Supplementary Rating Springs.

9. *The Rating Spring.*—On the opposite side of the pendulum to the counting mechanism is hinged a light horizontal rod, A B (Fig. 5), its distal end suspended by a fine wire to a

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* A light contact on the 'scape wheel, if adopted in place of this counting wheel, would absorb less energy, but it is doubtful whether such a contact would define the epoch with such a degree of accuracy. Such accuracy is essential for the close synchronising of the two pendulums.

cock, C, on a level with the cock of the pendulum. Below the horizontal rod, and attached at the same point as the suspension wire is a spiral spring, D E. The lower end of the spiral spring is continued by a watch fusee chain, which is wound round a barrel on the arbor of the star-wheel F. When the pendulum is at its vertical position the rod A B is horizontal and the suspension wire and spiral spring are vertical. An increase of the tension of the spiral spring brought about by turning the star-wheel F will increase the rate of the clock (§ 19); moreover, this mode of altering the clock's rate is free from appreciable backlash,* a point that will be found later (§ 13) to be of importance. A small weight, P, about equal to the tension of the spring at its mean value, hangs by a silk thread wound round the same barrel, and renders the star-wheel subject only to the small difference of the forces of the two.

10. *Supplementary Rating Spring.*—A rocking lever, G H, friction-tight on the arbor J of the star-wheel, is limited in its movements by the two stops K and L; a supplementary spiral spring, M G, parallel with the above, is attached by its upper end to the horizontal rod at M and by its lower end to the rocking lever at G. This second spring is counterpoised by an exactly similar spring attached by its upper end to the rocking lever and by its lower to a friction-tight arbor, N. This arbor can be turned by a milled head for the adjustment to equality of the tension of the two springs M G and G N, when the lever remains indifferently against either stop. The function of this supplementary spring will be explained later (§ 13).

11. *The Synchronising Mechanism.*—In §§ 4 and 6 two electric circuits were referred to—viz., (I.) that closed by the counting wheel, whereby the impulse lever of the master pendulum was released, and (II.) that closed by this lever on its fall, and whereby it was restored to its initial position. These two circuits pass also through electromagnets in the synchronising mechanism.

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* If, when the star-wheel, moving clockwise, reaches a certain position, the tension of the spring is the same as that attained when the star-wheel, moving counter-clockwise, reaches the same position, there is no backlash. In general, with such arrangements, the tension is less while the motion is in the direction to increase the tension. In the above arrangement the difference is very small.

12. *The Principle.*—The principle of the method of synchronising the slave clock with the master pendulum is as follows: Every minute the impulse lever is released at an instant dependent on the position of the slave pendulum. The lever rolls on the pallet and falls on its contact at a time dependent on the position of the master pendulum. There is an interval, *t*, of about ¼ second between the closing of circuit I. and circuit II. It is the function of the synchronising mechanism to maintain this interval constant. When the interval is greater than *t* the slave pendulum is in advance and its error is positive. Every minute a measurement of this interval is automatically made, when, if the error of the slave pendulum is positive, its rate is reduced by a constant amount, N; if negative, the rate is increased by the same quantity. These changes of rate are effected by altering the length of the rating spring (§ 9), and, therefore, its tension, by a constant quantity.

13. *False Synchronism.*—It is shown in § 20 that this method alone will result in a false synchronism, a hunting of error and rate, the error and rate passing through periodic variations of constant amplitude, the variation of the error following that of the rate with a difference of phase of a quarter of a period. Further, it is there shown that if there is any backlash of the rating mechanism, in the sense in which that word is used in § 9 Footnote, the amplitude of these periodic variations of error and rate will continually increase without limit. The function of the supplementary spring is to introduce a negative backlash, so to speak, which effectually reduces the amplitudes of these variations till they nearly disappear. When, by any change of conditions, the rate of the slave clock is altered, periodic variations of the error and rate are set up, which in course of time are damped out, and the rate of the slave pendulum readjusted to that of the master pendulum. In practice it is found that there is a continual wavering, of small amplitude, of the error and rate about a mean position,* this mean position continually adjusting itself to changing conditions.

14. *Description of Synchronising Mechanism.*—The electric current of circuit II. passes through the electromagnet A (Fig. 6). The armature B of this magnet is attached to the lever C (Fig. 6 and 7), which is held away from the poles of the

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* With the experimental model the amplitude of these variations of error rarely exceed ±0·005 (*see* Footnote to § 20).

magnet by the spring D. Attached to the end of this lever is the tongue E (Figs. 6, 7 and 8), movable about a vertical axis through a small distance of about twice its horizontal thickness determined by two stops (not seen in the figures) and sliding with a little friction on its guide. The end of this tongue is shown enlarged in Fig. 8. Engaging with the starwheel referred to in § 9, Figs. 5 and 7, are two pawls, G G (Figs. 7 and 8), movable about pivots, H H, and held against the stops J J by the springs K K. The points of these pawls when pressed upon the wheel F move it to the right or left, the movement through the space of one tooth being completed

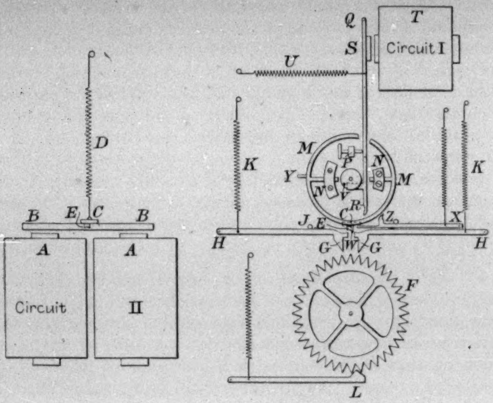


Fig. 6. Fig. 7.

The Synchronising Mechanism.

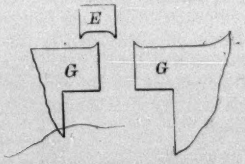
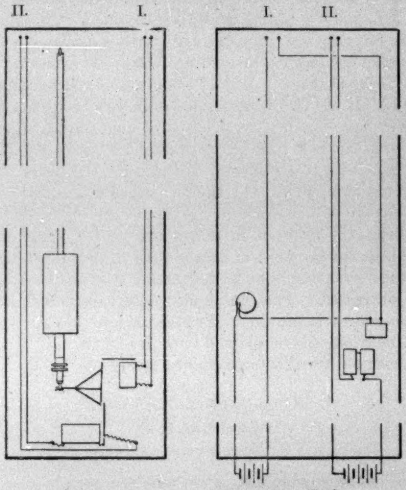


Fig. 8.

by the jumper L. The ends of these pawls are shaped as shown in Fig. 8.

When the lever C is drawn down by the electromagnet A and the tongue E is against its left stop, the latter engages with the left pawl, and the wheel is turned one tooth to the left; similarly, when the tongue is against its right stop the wheel is turned one tooth to the right. At intermediate positions of the tongue one or other of the pawls will generally be engaged; between the ends of the pawls there is but barely sufficient room to let the tongue pass.



Master Pendulum. Slave Clock.

FIG. 9.—THE ELECTRIC CONNECTIONS.

Above the star-wheel is the timing wheel M M (Fig. 7). This carries weights, N N, movable along a radius, whereby it is thrown out of balance, and these weights are so adjusted that the period of time the wheel takes to turn from rest through 90 deg. to its position of equilibrium is the period *t* referred to in § 12.

The lever Q R, pivoted at Q, attached to the armature S of the electromagnet T (circuit I.) holds the timing wheel by the tension of the spring U, and through the pressure against the ruby pin V, in a position 90 deg. from its position of equilibrium. The lever Q R is held against a screw stop, P. The pin W on the periphery of the wheel presses the tongue E to its left position. In this position the timing wheel is held during the intervals between successive minute impulses.

15. *Mode of Working.*—At the completion of each minute the counting wheel (Fig. 4) closes circuit I., the impulse lever is released on the pallet of the master pendulum, the lever Q R (Fig. 7) is drawn to the electromagnet, and is held by the detent X Z, and the timing wheel begins to fall. As this wheel turns the impulse lever runs down the impulse pallet. On falling off the pallet the impulse lever closes circuit II., is thrown back, and the lever C (Figs. 6 and 7) is drawn down, releasing the lever Q R. If the two pendulums are in synchrony the timing wheel will have reached its position of equilibrium when this occurs, and the pin Y on its periphery will have pushed the tongue to the right a sufficient distance for the latter to pass between the two pawls G G ; the star-wheel will not be turned, and no change will be made in the rate of the slave pendulum. If, however, the latter is in retard the timing wheel will be released late, the tongue will engage with the left pawl, and the rate of the slave pendulum be increased ; if in advance its rate will be decreased. This automatic comparison and adjustment is made each minute, and the synchrony of the two pendulums maintained as explained in §§ 12 and 13.

The electric connections are shown in Fig. 9.

III. MATHEMATICAL DISCUSSION.

Table of Symbols.

Quantities in C.G.S. units unless otherwise stated.

		Values in model.
$4a$	Latus rectum of parabola (figure of pallet)	1·40 cm.
E	Total energy of master pendulum	—
ΔE	Energy conveyed by each minute impulse	$1{\cdot}87 \times 10^3$ gm. cm.^{2} sec.$^{-2}$
$\Delta e'$	Change of daily rate, in seconds	—
F	Couple retarding pendulum by wheel friction on pallet	About $2{\cdot}4 \times 10^3$ gm. cm.^{2} sec.$^{-2}$
I	Moment of inertia of master pendulum......	$6{\cdot}47 \times 10^7$ gm. cm.^{2}
i	Moment of inertia of impulse lever..........	$2{\cdot}07 \times 10^2$ gm. cm.^{2}
K	Gravity couple of impulse lever...............	$3{\cdot}69 \times 10$g gm. cm.^{2}sec.$^{-2}$
L	Length of master pendulum, suspension to pallet	$1{\cdot}19 \times 10^2$ cm.
l	Length of impulse lever, pivots to centre of wheel	9·2 cm.
N	Change of rate in seconds per day imposed each minute	—

		Values in model.
P	Horizontal component of wheel pressure on pallet	—
s	Angular distance of centre of impulse wheel from beginning of impulse face when $\theta=0$; positive to right	—
T	Semi-period of pendulum	1 second
t	Time interval between the closing of the two electric circuits	—
V	Linear velocity of pallet when $\theta=0$	6·52 cm. sec.$^{-1}$
X	The quantity $\frac{1}{a}\frac{da}{dt}=\frac{1}{\log_{10}e}\cdot\frac{d\log_{10}a}{dt}$; $a=\frac{\pi}{180}$	1·60 × 10^{-4} sec.$^{-1}$
x_1	The horizontal travel of the pallet while the wheel is rolling down the impulse face...	8·3 × 10^{-1} cm.
a	Semi-amplitude of pendulum in radians...	$\pi/180$
θ	Position of pendulum at time under consideration; positive to the right.........	—

16. *The Figure of the Impulse Pallet.*—With a view to a steady fall of the impulse wheel, the impulse slope is so shaped that the path of the wheel relative to the pallet is a parabola. The wheel then falls with a constant acceleration.* The extent of the impulse is confined to about the middle fifth of the complete amplitude of the pendulum. Within these limits the velocity V of the pallet is nearly constant.

It can be shown that the horizontal component of the force exerted by the wheel on the pallet (*see* Fig. 10) is

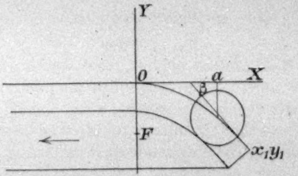


FIG. 10.—THE FIGURE OF THE IMPULSE PALLET.

$$P=\frac{x}{2al}\left(K-\frac{V^2i}{2al}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and the energy conveyed to the pendulum with each minute impulse is

$$\Delta E=\frac{x_1^{\,2}}{4al}\left(K-\frac{V^2i}{2al}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

---

* In order that the parabolic path may be strictly followed the radius of the wheel must not exceed a certain value dependent on the latus rectum and the constants K and i of the lever.

No special importance is attached to the above figuring of the impulse face. If d is the distance through which the wheel falls and β the angle with the horizontal made by the path of the wheel relative to the pendulum at the end of the impulse, the last equation can be written

$$\Delta E = \frac{Kd}{l} - \frac{V^2 i}{2l^2} \tan^2\beta,$$

and in this form is applicable to any figuring.

The first term is the energy of fall of the lever and the second the energy of motion with which the lever leaves the pallet.

The second term on the right of equation (2), involving the moment of inertia of the impulse lever, increases with the square of the amplitude. Hence the impulse conveyed to the pendulum is less when the arc is greater. The inertia of the lever, therefore, exercises some degree of control over the arc. As the writer is of opinion that it is advisable that the lever be as light as possible its inertia is made as small as is consistent with a sufficient value of K.

17. *The Energy Required to Maintain the Master Pendulum.*—The energy necessary to maintain the arc at the amplitude decided upon can be found by determining the decrement da/dt of arc when the pendulum is set to swing freely.

We have

$$\frac{da}{dt} = \frac{a\,d\log_e a}{dt} = \frac{a}{\log_{10} e} \cdot \frac{d\log_{10} a}{dt}.$$

Log $_{10}\,a$ is then plotted against time, and a graph obtained which is nearly linear. The tangent at the point corresponding to the amplitude decided upon gives $d\log_{10} a/dt$.

It can be shown that the energy required to be conveyed each minute to maintain the amplitude at a radians is

$$\Delta E = \frac{60}{\log_{10} e} I \frac{\pi^2}{T} a^2 \frac{d\log_{10} a}{dt} \quad . \ . \ . \ . \ . \quad (3)$$

Combining this with (2) we have

$$\frac{x_1^2}{4al}\left(K - \frac{V^2 i}{2al}\right) = \frac{60}{\log_{10} e} I \frac{\pi^2}{T} a^2 \frac{d\log_{10} a}{dt},$$

from which suitable values of K, i, &c., can be calculated that will give the required arc.

18. *The Couple Retarding the Pendulum Due to the Friction of the Wheel on the Pallet.*—A knowledge of the value of this couple is useful in order that its effects on the rate of the pendulum may be estimated (*see* § 27).

To determine this value the pendulum is swung with the wheel rolling under the weight of the lever on the horizontal portion of the pallet, which is made some centimetres long for this purpose. The decrement of amplitude with time is observed. Under these conditions this decrement is that due to the friction, in addition to that to which the free pendulum is subject. The latter has been found by the method of § 17, and the difference is that due to friction.

If F be the friction couple retarding the pendulum and Δa the change of amplitude due to wheel friction in time T, it can be shown that

$$\Delta a = -\frac{2FT^2}{I\pi^2}.$$

Within the limits of error of experiment, which are somewhat large, this couple is found to be independent of the amplitude.*

19. *The Rating Spring.*—In Fig. 11, A C is the slave pendulum, B E the horizontal connecting rod, D E the suspension wire attached to the rod at E, E F the spiral spring and its connections with the star-wheel. When the pendulum is vertical D, E and F are in a vertical straight line.

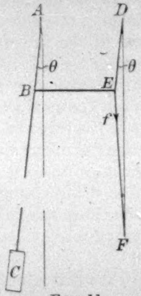


FIG. 11.

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* It may be of interest to refer to the changes in the value of F that were found with the experimental model under different conditions. When the wheel and its jewelled pivots were new and freshly lubricated F had the value $2{\cdot}13 \times 10$ grammes $\text{cm.}^2\,\text{sec.}^{-2}$. After a year when the pallet was dusty and the pivots not re-lubricated, this figure rose to 2·75; and after cleaning the pallet it fell to 2·39. The figure in the last decimal place is probably of little significance.

Let f be the tension of the spring and M the mass of the slave pendulum bob, and let $DE=AB=l_a$ and $EF=l_b$. Then, if ΔT be the change of period due to a change of tension Δf, we have with sufficient accuracy

$$\Delta T=-\frac{\pi^2\frac{l_a}{l_b}(l_a+l_b)}{2MgT}\Delta f.$$

From this the constants of the two springs to give the necessary change of period can be calculated.

20. The method of synchronising the slave clock with the master pendulum can be explained graphically. In Fig. 12, with the lower curves, ordinates represent rate imposed on the slave pendulum by the rating spring, zero rate being that attained when the two pendulums have the same period. Abscissæ represent time. Rate is imposed at each minute comparison in constant quantity N, positively when the error is negative, negatively when the error is positive. Curve I. represents the hypothetical case in which there is no backlash on reversal in the action of the mechanism altering the rate. In curve II. there is backlash of value $2N$, as shown by the dotted line following behind the full line when a change of direction takes place. In curve III. a negative backlash of value $2N$ imposed by the supplementary spring is shown by the dotted line preceding the full line on change of direction.

The full line represents in the reversed direction the position of the lower end of the principal rating spring.

In each case the initial state ($t=0$) is taken when error$=0$, rate$=5\,N\pm$backlash. The error accumulated (Nt) is represented by the area between the graph and the zero line, changing positively when the graph is above, negatively when below the line.

In curve I., at the end of the first minute, rate being positive, a positive error has accumulated, and consequently the rate falls, and in five minutes a positive error represented by the area a, b, c has been acquired. The rate continues to fall until the positive error has been wiped out, which, from the symmetry of the figure, obviously occurs when it has reached a negative value equal to the positive value from which it started. This hunting of error and rate with constant amplitudes about their zeros, with error a quarter of a period in phase behind rate, will continue indefinitely until a disturbance increases or decreases the amplitude.

In curve II., owing to backlash, the real rate imposed,

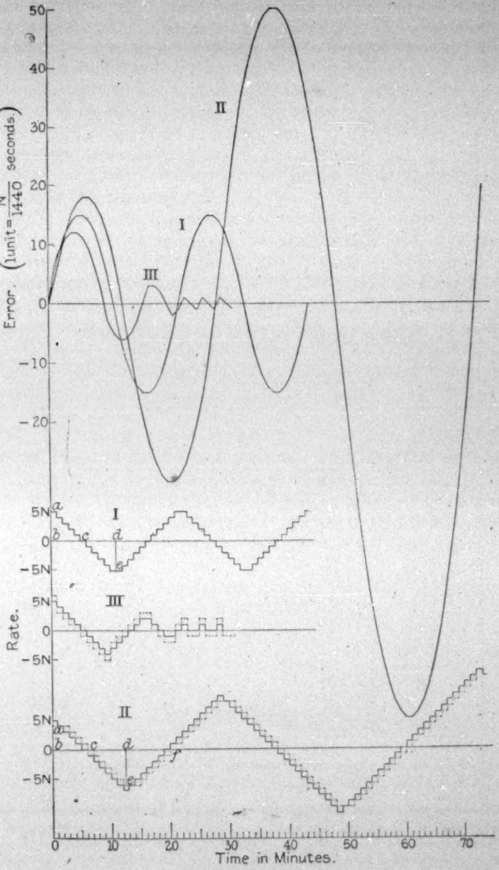
50
40
30
20
10
0
-10
-20
Error. (1unit = N/1440 seconds.)
I
II
III
5N
0
-5N
a
b
c
d
e
I
5N
0
-5N
III
Rate.
5N
0
-5N
II
a
b
c
d
e
f
0
10
20
30
40
50
60
70
Time in Minutes.

FIG. 12.

shown by the dotted line, follows behind the action of the mechanism. Consequently a greater fall in the rate takes place before the area *c*, *d*, *e* equals the area *a*, *b*, *c*—that is,

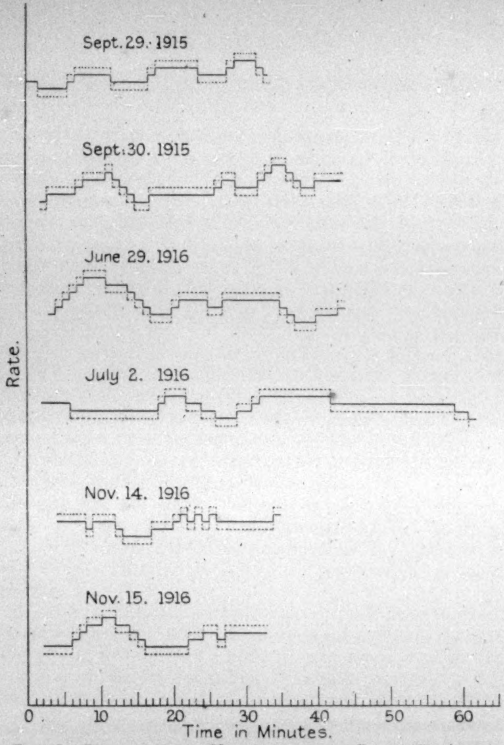

Fig. 13.—Records of the Movements of the Rating Springs of the Experimental Model.

before the negative rate has wiped out the accumulated positive error. The area *d*, *e*, *f* is again larger for the same reason.

In this way the amplitudes and the periods of the swings grow larger indefinitely.

Curve III. shows the opposite effect of negative backlash due to the supplementary spring, the amplitudes and periods of the changes in rate and error becoming less until they are reduced to small waverings on alternate sides of the zero. These small changes never cease, but the accumulated error at any time is inconsiderable.

The same three cases are plotted in the upper curves, but with ordinates denoting error.

In Fig. 13 are represented some actual records of the movements of the rating spring of the experimental model. Here N is about $\frac{1}{3}$ second per day and the movement of the supplementary spring imposes a rate of $2N$. The zero line changes with changing conditions of the slave clock. There is a small unknown margin of error within which no change of rate takes place. There are therefore periods of varying length during which no change of position of the rating spring is made.*

21. The timing wheel, weighted at one side, is held at 90 deg. from its equilibrium position. It is adjusted so that the time taken in falling through 90 deg is the interval t between the closing of the two electric circuits (*see* §§ 12 and 14). Let I_ω be the moment of inertia of the wheel and $K_\omega \sin\theta$ its restoring couple when displaced θ from the vertical.

The time occupied in falling through θ to the equilibrium position is

$$t=\frac{\pi}{2}\sqrt{\frac{I_\omega}{K_\omega}}\left\{1+(\tfrac{1}{2})^2\sin^2\frac{\theta}{2}+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4\frac{\theta}{2}+\&c.\right\} \quad . \qquad (4)$$

Putting

$$\theta=\frac{\pi}{2}, \text{ we have } t=1{\cdot}85\sqrt{\frac{I_\omega}{K_\omega}}.$$

---

* With the experimental model the accumulations of error rarely, if ever, amount to ±1/200 second. The slave clock used has a poor train, and its pendulum shows periodic changes of amplitude; moreover, the synchronising mechanism is somewhat crude, the parts having been designed by estimation and never subsequently altered; also the value of N and the unit of negative backlash have been made greater than has proved to be necessary. With a better slave clock, having a heavier pendulum, and with improvements in the synchronising mechanism learned by experience to be advisable, the accumulations of error could very probably be made never to exceed ±1/1000 second.

The angular velocity on reaching the equilibrium position is

$$\sqrt{\frac{2K_\omega}{I_\omega}}=\frac{2\cdot 62}{t}.*$$

From (4) we have—

$$\frac{dt}{d\theta}=0\cdot 27\frac{\pi}{2}\sqrt{\frac{I_\omega}{K_\omega}}=0\cdot 27\frac{\pi}{2}\,\frac{t}{1\cdot 85}.$$

Hence, if t is $\frac{1}{4}$ second, a variation of the initial position of the wheel of 1 deg. will cause a variation of the time of fall of only 1/1000 of a second, a point of practical importance.

IV. DISCUSSION OF CERTAIN CAUSES OF VARIATION AND THEIR PROBABLE EFFECTS ON THE RATE.

22. The effect of change of temperature acting directly on the rate of the master pendulum is not considered, as it is assumed that this has been correctly compensated in the pendulum, and that the uniform conditions as regards temperature to which the pendulum is intended to be exposed will render these negligible.

Neither is any consideration given to change of atmospheric density, this, it is assumed, having been eliminated by the enclosing of the pendulum in an airtight case.

The variations requiring consideration seem to be comprised in the following :—Changes in

(*a*) The friction of the wheel on the impulse pallet.

(*b*) The instant of release of the lever on the pallet.

(*c*) The relative position of the pallet and the impulse lever due to tilting sideways of the master pendulum and its mechanism and of the sustaining wall.

(*d*) The conditions of the impulse lever.

(*e*) The effect of the blow of the wheel on the impulse pallet.

These variations may generally affect the rate in two distinct ways, which can be considered separately—viz., (i.) by varying the amplitude and so altering the rate by circular error, and (ii.) by varying the disturbance to the free swing during contact between the wheel and the pallet and so introducing escapement error.*

---

* In the experimental model the radius of the wheel at the point where the pin engages with the tongue (*c*, Fig. 7) is 1·7 cm., and t is about $\frac{1}{4}$ second, hence, the velocity acquired is 18 cm. per second. The dead space within which the tongue can move without engaging either the accelerating or retarding pawl is less than $\frac{1}{3}$ mm., so that the dead interval of time is less than 1/500 second.

23. *Circular Error.*—There is some reason to doubt the strict applicability in practice of the theoretical formula. In the first place a pendulum suspended by a spring does not exactly follow a circular path, and, secondly, with change of amplitude there result changes of non-conservative resistances which probably slightly affect the period. In default of better information we will calculate on the lines of the elementary theory.

Consideration of circular error may be resolved into consideration of changes of total energy of the pendulum.

The semi-period of a pendulum is

$$T=(1+\tfrac{1}{4}\sin^2\frac{a}{2}+, \text{\&c.})\pi\sqrt{\frac{l}{g}}.$$

Hence, with sufficient accuracy

$$dT=\frac{T}{8}ada. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

We have seen (§ 17) in dealing with the decrement of amplitude with time due to air-resistance that the expression

$$\frac{1}{a}\frac{da}{dt}=\frac{1}{\log_{10}e}\cdot\frac{d\log_{10}a}{dt}$$

is nearly constant, and a method of determining this was given.

Calling this quantity X, we have from (3)

$$\Delta E=60I\frac{\pi^2}{T}Xa^2.$$

Hence $$d(\Delta E)=120I\frac{\pi^2}{T}Xada. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Hence, from (5) and (6) a small change $\Delta'E$ in the energy conveyed each minute effects a small change ΔT in the period expressed by

$$\Delta T=\frac{T^2}{960I\pi^2X}\Delta'E,$$

or $$\frac{\Delta T}{T}=\frac{a^2}{16}\frac{\Delta'E}{\Delta E}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

---

* This term is used as a convenient one for change of rate due to change of the disturbance of the free swing, though in the present case there is no escapement.

And the change in the rate due thereto is—

$$\Delta e' = -86{,}400\,\Delta T \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (8)$$

in seconds per day.

24. *Escapement Error.*—The loss or gain of time ΔT in a single vibration due to a disturbance at a point θ, thereby changing the energy of the pendulum by $\Delta' E$ can be shown to be expressed by

$$\Delta T = \pm \frac{T^3}{I\pi^3} \frac{\theta}{a^2\sqrt{a^2-\theta^2}} \Delta' E,^* \quad . \; . \; . \; . \; . \quad (9)$$

the positive sign applying to the case where the disturbance occurs when the pendulum is rising, the negative when falling, θ being positive.

If this change of period occurs with each minute impulse, the change of rate due thereto is

$$\Delta e' = -1{,}440\,\Delta T \quad . \; . \; . \; . \; . \; . \quad (10)$$

in seconds per day.

25. The application of the above expression for escapement error to friction of the impulse wheel is of importance and will be required in the discussion to follow.

From the definition of the frictional retarding couple (§ 18)

$$dE = \mp F d\theta,$$

the upper sign applying when the pendulum is rising and θ is positive, whence from (9)

$$dT = -\frac{T^3 F}{\pi^3 I} \frac{\theta}{a^2\sqrt{a^2-\theta^2}} d\theta.$$

In integrating this expression it is to be noted that both a and θ are variable. With such values as F is likely to have the error involved in treating a as constant is negligible. The change of period of an individual vibration due to friction of the wheel between θ_0 and θ_1 is then

$$\Delta T = \frac{T^3 F}{\pi^3 I a^2} \{\sqrt{a^2-\theta_1{}^2} - \sqrt{a^2-\theta_0{}^2}\} \quad . \; . \; . \quad (11)$$

26. The application of the same expression (9) to the disturbance of free-swing brought about by the impulse will also be required. This will depend on the position of the pendulum at which the impulse is adjusted to take place.

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* *See* G. B. Airy, "On the Disturbances of Pendulums, &c.," Camb. Phil. Soc. Trans., Vol. III., 1830, p. 105; J. M. Bloxam, Memoirs A.S., Vol. XXII., 1853.

In Fig. 10 let the position of the wheel as drawn represent that which it occupies on the pallet when the pendulum is at its vertical position. The pallet is moving to the left, and the wheel is rolling down the impulse face from $x=0$ to $x=x_1$. Let s be the angular distance $0a$.

It can be shown from (1) and (9), the positive direction of θ being taken towards the right, that the change of the individual period due to the element of impulse through $d\theta$ is

$$dT=\frac{2T^3L^2}{\pi^3 I}\frac{\Delta E}{x_1^{\,2}}\frac{\theta(s-\theta)d\theta}{a^2\sqrt{a^2-\theta^2}},$$

and the whole change of period due to the impulse, treating a as constant, is

$$\Delta T=\frac{T^3L^2}{\pi^3 I}\frac{\Delta E}{x_1^{\,2}}\cdot\frac{1}{a^2}\left\{a^2\sin^{-1}\frac{s}{a}-a^2\sin^{-1}\frac{s-\frac{x_1}{L}}{a}\right.$$
$$\left.+s\sqrt{a^2-s^2}-\left(s+\frac{x_1}{L}\right)\sqrt{a^2-\left(s-\frac{x_1}{L}\right)^2}\right\} \quad . \quad . \quad (12)$$

Owing to the greater forces acting in this case the discrepancy involved in assuming the constancy of a is appreciable. In the application in § 30 the value of ΔT obtained from the use of this expression was found to differ by about 2 per cent. from the rigid value.

27. We are now in a position to consider the effect on the rate of the clock of the changes set out in § 22. In order to do this the data supplied by the experimental model will be used (*see* Table of Symbols).

(*a*) *A Change in the Value of the Friction of the Wheel on the Impulse Pallet.*—(i.) *Circular Error.*—Assuming that friction is proportional to the varying pressure, and neglecting the inertia term in (1), we find (*see* §§ 18 and 25) the change of energy of the pendulum due to friction during one impulse is

$$\Delta_2 E=F(\theta_3-\theta_1),$$

where θ_1 is the position when the lever falls on, and θ_3 when it falls off, the pallet.

Hence, the change due to change of friction is

$$\Delta' E=\Delta F(\theta_3-\theta_1).$$

And from (7) and (8) the change of daily rate is

$$\Delta e'=-86{,}400\frac{Ta^2}{16\Delta E}\Delta F(\theta_3-\theta_1).$$

Putting $\theta_1=0{\cdot}3a$, and $\theta_3=-0{\cdot}25a$, as in the adjustment made in the model; and putting $\Delta F=6{\cdot}2\times10^2$, probably an extreme value (*see* Footnote to § 18), we find

$$\Delta e'=+0{\cdot}0052 \text{ second per day.}$$

(ii.) *Escapement Error.*—Using the same value for the change of friction, we have from (10) and (11)

$$\Delta e'=-0{\cdot}00036 \text{ second per day.}$$

28. (*b*) *A Change in the Instant of Release of the Impulse Lever.*—Such changes inevitably occur by the nature of the mode of synchronising, but they are periodic, and their effects are not cumulative. Probably an extreme value of such a change, taking one whole day with another, would be ±1/100 second. The normal value $\theta_1=0{\cdot}3a$ becomes $\theta_1=0{\cdot}3a\pm0{\cdot}03a$.

This change may conceivably affect the rate by (a) varying the length of pallet along which the wheel rolls, whence by (i.) circular error

$$\Delta e'=-0{\cdot}0011 \text{ second per day,}$$

and by (ii.) escapement error

$$\Delta e'=+0{\cdot}0009 \text{ second per day.}$$

(β) This change may also affect the rate by altering the position in the arc at which energy is drafted by starting the wheel or by the blow on the pallet. The effect of the former is negligible, and that of the latter is dealt with in § 31.

29. (*c*) *A Shift of the Point of Suspension Sideways Relatively to the Impulse Lever.*—Such a shift or tilt might arise from a want of stability of the wall to which the master pendulum was attached, or from a faulty attachment of the pendulum or the impulse mechanism to such wall.

(a) The immediate effect will be to alter the value of s (*see* § 26). Differentiating the expression (12) with respect to s, we get by escapement error

$$d(\Delta T)=\frac{2T^3L^2\Delta E}{\pi^3Ix_1{}^2a^2}\left\{\sqrt{a^2-s^2}-\frac{a^2-s^2+\frac{x_1}{L}s}{\sqrt{a^2-\left(s-\frac{x_1}{L}\right)^2}}\right\}ds \quad . \qquad (13)$$

Putting, as in the model, $s=0{\cdot}15a$, and giving ds the value 1′ of arc, we find

$$\Delta e'=0{\cdot}076 \text{ second per day.}$$

This is a large variation; moreover, since $d^2\Delta T/ds^2$ is very small with all possible values of s, this figure cannot be materially reduced by altering the position in the arc at which the impulse is given.

This investigation points to the necessity of firm attachment of the pendulum and its mechanism to their support. The same effect would obtain with the Graham dead-beat and the generality of escapements, but with those where the impulse is given symmetrically in two directions the effect in one direction is compensated by the counter effect in the other when the tilt is small.

This effect is not a consequence of the particular way in which the impulse is given. The most ideal impulse where all the energy is given at the position $\theta=0$ would yield an almost identical figure.

(β) Another effect of such a tilt will be one similar to that dealt with in § 28 (*b*), but the changes of rate are smaller than there resulting.

30. (*d*) *A Change in the Conditions of the Impulse Lever.*—(*a*) One of the conditions here to be considered is that of friction at the pivots. Such friction will affect the rate by altering the value of the impulse.

In order to gain some knowledge of the pivot friction a simple experiment was made. The pivots and their holes had not been cleaned for about two years. The static friction was found to amount to about $K/400$.

(i.) *Circular Error.*—We have from (2), neglecting the inertia term,

$$\frac{\Delta' E}{\Delta E}=\frac{\Delta K}{K},$$

and from (7) and (8)

$$\Delta e'=-86{,}400\frac{Ta^2}{16}\,\frac{\Delta K}{K}.$$

Assuming an increase of friction amounting to 100 per cent. of its value, we get

$$\Delta e'=+0{\cdot}0041 \text{ second per day.}$$

(ii.) *Escapement Error.*—Applying the formula (12), putting $s=0{\cdot}15a$, we get, with the same increase of friction,

$$\Delta e'=+0{\cdot}0013 \text{ second per day.}$$

It may be pointed out that by altering the position of the pendulum at which the impulse is given the disturbance to the pendulum by the impulse may be reduced or made to vanish, and changes of this disturbance with change of impulse to vanish also. Thus, if $s=0{\cdot}27a$, ΔT of expression (12) becomes zero. As, however, the absolute value of this disturbance is of no account if it remains constant, as the probable variations of the impulse are small, and as a higher value of s involves the pushing of the point at which the lever falls further from the zero position, it has been judged advisable to give s the value $0{\cdot}15a$.

(β) Another condition of the impulse lever is that of temperature. It is assumed (*see* Introduction) that the changes of temperature to which the master pendulum are exposed are small. A small change of temperature, however, may have a serious effect if it alters the position in the swing at which the impulse is made. Suppose, for instance, that the pivots of the lever remain in constant relation to the pallet while the steel lever expands, then s will change by the quantity

$$\Delta s \doteq -\frac{l}{L} \times 0{\cdot}000011 \times 5$$

for an increase of 5°C., putting 0·000011 as the coefficient of expansion of steel.

From (13) and (10) the change of rate will be

$$\Delta e' = -0{\cdot}0011 \text{ second per day.}$$

31. (*e*). *A Change in the Effect of the Fall of the Wheel on the Impulse Pallet.*—In order that the blow of the wheel on the pallet shall be small it is essential that the wheel shall be held but a fraction of a millimetre above the pallet. This involves the rigid attachment of the impulse mechanism to the suspension. It is proposed to suspend the pendulum in a cast-iron box with a plate-glass window, and to attach the impulse mechanism to the cast iron. The relative change of vertical position of the wheel and pallet due to expansion through 5°C. would be about 0·0055 cm.

Careful consideration has been given to the question whether the impulse should not preferably be given at a point higher up the pendulum with a view to diminishing any relative change in position of wheel and pallet. It appears that any advantages of so doing are outweighed by the difficulties and disadavantages.

With a view to measure the effect of the blow a series of experiments were made, in each of which during a measured time (20 minutes) the wheel was dropped a large number of times (337 to 400) through a distance of 0·02 cm., and, after a short measured run along the horizontal pallet, was raised. The decrement of amplitude was read. The experiments were rough, but within the probable limits of error the decrement of amplitude was fairly closely accounted for by the air resistance and the frictional resistance of the run. The resistance to the pendulum due to the blow of the wheel seemed very small.

Table of Resulting Changes of Rate.

Due to—	By circular error.	By escapement error.
	Second	per day.
(*a*) Change of friction of impulse wheel (25 p.c.)...	±0·0052	∓0·00036
(*b*) Change of instant of release (0·01 second) of impulse lever	±0·0011	∓0·0009
(*c*) Tilt of master pendulum attachments (1′ of arc)	...	±0·076
(*d*) (α) Change of friction at impulse lever pivots (100 p.c.)..	±0·0041	±0·0013
(β) Expansion or contraction of impulse lever through 5°C.	...	∓0·0011

The change in each case is taken as acting through one whole day.

It may appear that in the course of this discussion of probable causes of variation of rate, the mathematical mill has been applied to grind out results of greater refinement than is consistent with the trustworthiness of the data that can be used. Considering, however, the simplicity of the conditions to which this pendulum is exposed, this mathematical application seems to the writer to be justified; he believes that these results give useful evidence of the order of magnitude of the variation of rate to be expected from the various causes considered. It is possible that other causes of variation that have not occurred to him may be found to be of equal or greater importance.

32. *The Best Value for the Amplitude of the Master Pendulum.*—The degree of freedom from frictional disturbance and the confining of the impulse and all mechanical contact to a small angle on each side of the vertical position lead one to infer that escapement errors will be of less importance as compared with circular errors in the case of this clock than usually obtains, and this consideration seems to justify the adoption of the unusually small semi-amplitude of 1 deg.

The following table has been calculated for a semi-amplitude of 2 deg., certain reasonable and convenient necessary modifications of other conditions having been made. A comparison of this table with that of § 31 will illustrate the argument used above.

It is anticipated that the error due to the change (*c*) can be reduced to small value :—

Table of Changes of Rate with Semi-amplitude doubled (2 deg.).

Due to—	By circular error.	By escapement error.
	Second	per day.
(*a*) Change of friction of impulse wheel (25 p.c.)...	±0·006	∓0·0002
(*b*) Change of instant of release of impulse lever (0·01 second)	±0·002	∓0·0003
(*c*) Tilt of master pendulum attachments (1′ of arc)	...	±0·038
(*d*) (*a*) Change of friction at impulse lever pivots (100 p.c.)..	±0·016	±0·0006
(*β*) Expansion or contraction of impulse lever through 5°C.	...	∓0·0006

V. CONCLUSION.

It is unusual to put forward a new piece of physical apparatus without any attempt to report as to its efficiency in practice. But in this case the conditions are unusual. No comparison of a clock of the precision to which this aims is of any use except with the daily records of a first class observatory. The Paris rhythmic time signals have not been available to private observers during the war. Moreover, the purpose of the experimental model is merely to show the practicability of the method of synchronising the two pendulums and of the mode of giving the impulse to the master pendulum ; to show that the clock works without risk of failure of the electric contacts or other details. This the writer claims has been achieved. He suggests that, the practicability of the method being shown, the conditions for good timekeeping are assured.

The model exhibited has involved considerable expenditure, and the writer feels justified in bringing the subject forward in its present stage in the hope that he may be sufficiently encouraged to expend greater sums on an instrument capable of undergoing the severest tests.

Warm acknowledgment is due by the writer to Mr. Joseph Kirner for the skilful way he has carried out the ideas embodied in this piece of apparatus, for the many valuable practical suggestions he had made, and for the great interest he has shown.

The writer is much indebted to Prof. Alfred W. Porter for kindly reading the manuscript, and to Prof. C. V. Boys, who has read the Paper and advised as to its publication.

ABSTRACT.

The principal feature of the author's arrangement is the employment of a " slave " clock to do the great part of the work, leaving the master pendulum no function beyond that of controlling the rate of the other. The master pendulum swings freely except for a short period (about one-fifth of a second) every minute, during which it receives an impulse from a small falling wheel electromagnetically released by the slave clock. At the end of its fall the impulse mechanism closes a second circuit and is restored to its initial position. These two electric circuits also energise parts of the mechanism in the slave clock by which the latter is kept in time with the master pendulum. The lagging of correction behind error, with the resulting periodic fluctuation in the rate, is reduced almost to the vanishing point by the introduction of a "negative backlash" in the control mechanism.

The Paper contains a mathematical discussion of the best working conditions and of the possible magnitude of errors which might arise from various causes.

DISCUSSION.

Prof. Boys contributed the following remarks : The subject with which Mr. Bartrum has dealt with such ingenuity and success is one which has the greatest fascination for me, and I am sure that the members of the Society will join with me in congratulating him. I feel that we should welcome this contribution the more because the author's daily work is quite in another walk of life. He has referred to the late Sir David Gill's clock with two pendulums, which I saw when I was at Capetown, and which Sir David explained to me very fully. While admiring the ingenuity of this, I felt that it was not a convenient solution of the general problem. In particular I distrusted the use of a pendulum hooked up and liberated every two seconds, so that the slave clock, in fact, was started and stopped 30 times a minute by the master pendulum. Mr. Bartrum has undoubtedly worked on safer and more practicable lines in employing for the slave clock a complete going clock which is only rated instead of being kept alive by the master pendulum, and the less frequent interactions, which are sufficient, are certainly much more desirable. Of the numerous elements which go to make up the complete device, I think that which I most admire is the device for negative back lash, and the short account of this and the purpose that it serves would I think, be read with advantage by anyone who has to do with governors of any kind, and of the difficulties due to hunting which are met with in such varied directions. The author has realised with very great perfection the ideal conditions of a free pendulum which may be placed 60 ft. below the ground in a place of uniform temperature and within a casing, so as to be protected from variations of atmospheric density, whether due to change of temperature, if any, or of barometric pressure. Such a pendulum, free from mechanical friction variations, and with a gravity impulse related exactly as wished to the centre position, should give the most perfect time keeping that can be realised. At the end of the Paper the author refers to the question of the most suitable arc of vibration. It is possible that this can only be determined experimentally, as it has been with escapement clocks, for there are here as elsewhere the counteracting advantages of improving isochronism with reduction of arc and improving stability of amplitude with increase of arc. However, such tests could only be made usefully under the most perfect conditions for the master pendulum, which up to the present the author has not been able to realise. He has, however, shown that his device will work, and that one pendulum will govern that of the other clock for some two years without trouble, and I consider that he has proved his proposition. This is not a mere paper invention full of ingenuity which, to use a common but erroneous expression, is all right in theory but in practice won't work. In practice it does work, and it does not appear to me to be as a corollary all wrong in theory. When any inventor breaks up new ground and brings to light new material of value he is apt to be shadowed by a host of minor inventors who very possibly may make some improvements in detail. In the present case it is certain that the comparatively small number of people who are interested in the production of the perfect clock will see other ways of doing what Mr. Bartrum has done. In fact, each one, if he thinks about it, will probably devise two or three different ways, and among all these, bits here and bits there, may very well be combined to produce a more convenient design. But there is no particular merit in doing this; the merit is in the device which others seek to modify rather than in the modifications. It is rare that there is so much merit in a parody as in an original composition. I wish to lay stress on this, because after seeing Mr. Bartrum's clock I found it impossible to avoid thinking of other devices. These were mainly due to features in Mr. Bartrum's design which I did not quite like. For instance, that against which my mechanical sense most rebelled was the labour imposed upon the slave pendulum of dragging round the timing wheel. I also disliked the idea of the springs and connections to the pendulum of the regulating gear. I feel that these impose such a burden on the slave pendulum as possibly to make regulation far more frequently needed than it would be if this pendulum were as untramelled as that of an ordinary regulator. In other words, that the pendulum is truly described as a slave rather than as a colleague. I wanted, therefore, to see if the regulator clock could not be left just as free and undisturbed as any regulator, for in that case the problem of governing it should be simplified. For instance, there are 86,400 seconds in a day, and an uncertainty of one second a day should not be possible in any such type of clock. A correction of rate of 1/80,000 in round figures is all that can ever be needed to keep the colleague clock in order. Such a correction is effected by attaching to the pendulum rod half way down to the centre of oscillation a weight of 1/10,000 of that of the bob. In any one minute the cumulated effect of such a change of rate is about 1/1,400 second. If, therefore, a bicycle ball of convenient weight can be tipped into a hole in a shelf half way down or lifted out again we have sufficient correction for any possible variation of rate, and with this alone, which is Mr. Bartrum's negative back lash, and no other correction at all, the necessary governing can be done, and never a 1,000th of a second divergence arise between the pendulums.

In the master pendulum case, which it ought not to be necessary to visit more often perhaps than once in 20 years, there should be no pivots and no lubrication. The clock maker dotes on pivots, and here I think Mr. Bartrum has come unduly under his influence. I think the impulse could better be given by a weight resting upon an arm carried by and one with the pendulum on one side of the suspension spring, with an effective fall of precisely limited extent, and operating once a minute. An electromagnetic device for holding this weight out of the way or for placing it in a defined position, so that it may be picked up by the rising arm and arrested in a lower position from the falling arm is easily designed, and if here a pivot should be found convenient it need not involve lubrication.

The timing wheel and its operation is one of Mr. Bartrum's devices which I admire, but, together with the counting wheel, it can be abolished altogether if the two pendulums are made synchronous in phase and kept so. In order to make such a method possible I would suggest that the escapement wheel of the colleague clock should make a relay contact when the clock ticks which should cause a spring supported plate below each pendulum to rise into operative position and there be held for about one second. Each of these carrying a rod of Acheson graphite would then make contact with fine wires carried at the ends of the pendulums just as they pass their middle positions. These two contacts would one of them send a current to a racing relay if the other had not sent one first. Whichever one, therefore, won the race, and got in first, would cause the relay to send on the appropriate one of two currents, one to put the ball on the regulating shelf of the colleague pendulum and the other to take it off. The colleague clock would also make a contact in the pendulum plate in plenty of time to cause the impulse weight to be put in its place ready to be picked up by the master pendulum, and another one to elevate it out of reach when it had done its work. It seems quite unnecessary to describe the very simple electromagnetic devices needed.

I should put a second pendulum plate near the end of the swing of the master pendulum, which the operator could lift for a couple of seconds once a day or once a month as found desirable, so that four successive contacts could be recorded on a chronograph from which the amplitude could at once be determined. I should also bring up two pipes from the bottom and the top of the master pendulum case to the observatory, where a pressure gauge would show the pressure, and so that if it were desired to regulate to a very small extent the rate of the master pendulum this could be done by small variations of pressure, or better by passing to the bottom of the case a known quantity of dry hydrogen gas, or to the top of the case dry heavy gas such as CO_2, so as to alter the density of the surrounding medium to a known but small extent. In normal times I could have made in a few days all that would be necessary to put these suggested modifications of Mr. Bartrum's invention into practice, and I should certainly have done so, as the problem is to me one of very great interest, but it is not possible now to find time for such excursions, and so I have put these forward as suggestions only. I had hoped to have been able to make some experiments in contacts between Acheson graphite and tungsten or tantalum wires, using a number of specimens of these wires, which the British Thomson-Houston Company have kindly given to me for this purpose, but at present this has not been possible. However, in these days contact difficulties are not what they were, if, indeed, they exist at all. The automatic telephone exchange and many other things would wholly fail if contacts were the bogey that they used to be considered.

Mr. **Duddell** said that at the Paris Observatory there was a standard seconds pendulum in a chamber 60 ft. below the surface of the ground. The temperature was constant, and there were no pressure variations, as the pendulum was in a vacuum. It was driven electrically, and maintained a perfectly constant rate, requiring no attention whatever. It gave a signal in a telephone every second.

The **Author**, in a communicated reply, said that he agreed that only experience could show the best amplitude to adopt. The President had pointed out that if the time-keeping conditions of the auxiliary clock were so good that its possible variations were small, the control of the master pendulum might be limited to the imposition of two rates, respectively above and below the true rate. This, the author thinks, contains a valuable suggestion. He points out that the master having such limited influence over its colleague, these rates would require to be carefully selected to avoid the danger of the colleague becoming beyond control. The further suggestion that the two pendulums should be maintained without difference of phase involves an electric contact being made by the master, an arrangement the author had been anxious to avoid from fear of its probable variability. If modern materials would enable such a contact to be made safely important simplifications could be effected.

VIII. *The Effect of the Water-vapour in the Atmosphere on the Propagation of Electromagnetic Waves.* *By* FREDERICK SCHWERS, *D.Sc.*

RECEIVED OCTOBER 30, 1916.

I PROPOSE in the following pages to discuss a point in connection with the propagation of electromagnetic waves in space. It seems well founded that the propagation of these waves is strongly influenced by the conditions of ionisation of the air; yet a quantitative theory has not been established since the actual facts—namely the change of intensity at the receiver from day to night, during solar eclipses, &c.—are not known with sufficient accuracy.* For the present, however, let us disregard the effect of ionisation and consider only the refraction of these waves through the atmosphere.

Suppose our sending and receiving stations are on the opposite shores of an ocean, so that they are separated by salt water only; in this case, the energy sent from an antenna, as shown by Sommerfeld,† is propagated for the larger part in the form of space waves, and only in smaller part in the form of surface waves.

Kiebitz ‡ has made an attempt to calculate the influence of the atmosphere in refracting the space waves towards the surface of the earth. Considering a layer of air 1 km. thick, he found that the refraction produced in this medium of feeble dielectric constant is quite insufficient to bring all the spacial waves down to the surface of the earth; it would thus appear that refraction through the medium air exerts but a small favourable influence on the intensity at the receiving antenna. Considering further the influence of moisture, Kiebitz found that it only increases the dielectric constant by 10 per cent., *i.e.* only to a small extent. In this last respect the calculation of Kiebitz seems to me quite erroneous. Without considering the experimental figures for the dielectric constant of water-vapour, he simply assumed that the expression of Clausius-Mosotti $\frac{k-1}{k+2}\cdot\frac{1}{d}$ remains constant when passing from the liquid to the gaseous state. If we put for liquid water at $T=293^\circ$ $k=81$, we find for water-vapour at 293° and 760 mm. the value $k=1{\cdot}0022$, and with this figure Kiebitz made his further calculations.

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* *See* E. G. Eccles, "Jahrb. f. Drahtl. Telegr.," Vol. VII., p. 191 (1913).

† "Annalen der Physik" (IV.), Vol. XXVIII., p. 665 (1909).

‡ "Jahrb. f. Drahtl. Telegr.," Vol. VII., p. 154 (1913).

As a matter of fact, it can easily be shown by a few examples that the Clausius-Mosotti formula does not remain constant when passing to another state of aggregation of a substance. For the gaseous state particularly, we find very high values of $A\left(=\frac{k-1}{k+2}\frac{1}{d}\right)$, especially when the dielectric constant in the liquid state is already very high. The following table gives a few figures in this respect* :—

TABLE I.

—	*T.*	*A* liquid	*T.*	*A* gas
	Deg.		Deg.	
Ethyl alcohol	287·7	1·12	383·0	1·47 (Baedeker)†
Acetone	185·4	1·01	...	...
	273·0	1·06	290·3	3·54 (Pohrt) ‡
	291·8	1·12	382·3	2·77 ,,
Ammonia	288·0	1·35	291·4	3·40 (Baedeker)
	...	...	292·0	3·29 ,,
	...	...	332·4	2·91 ,,
	...	...	335·1	2·88 ,,
	...	...	356·8	2·75 ,,
	...	...	368·3	2·67 ,,
	...	...	381·4	2·65 ,,
Water	273·0	1·03	413·0	4·54 (Baedeker)
	283·0	0·98	415·2	4·57 ,,
	...	...	416·2	4·39 ,,
	...	...	418·8	4·16 ,,
	...	...	421·6	3·90 ,,

The *A*-values for water and ammonia in the gaseous state are plotted on diagrams I. and II. Ammonia is interesting as compared with water, since the values extend over a much wider range of temperature. Baedeker states that in both cases experiments could not be made at lower temperatures because of the conductivity of the gas ; in fact, we notice that for water as well as for ammonia the two first values are somewhat off the general curve.

If for water-vapour we extrapolate down to $T=293°$, by assuming the straight line given by the diagram, we arrive at $A=16·45$, which gives $k=1·037$ for 760 mm., thus much more than the previously calculated figure 1·0022. It must be admitted that such an extrapolation is hazardous ; there is every reason to believe that the temperature function of A runs, not according to a straight line, but according to a curve, the convexity of which is turned towards the abscissa, as in the case of ammonia vapour. For the latter $\frac{dA}{dT}$ increases from 0·005

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* We consider the values of k for large λ, thus practically for $\lambda\infty$.

† "Zeitschr. Phys. Chemie.," Vol. XXXVI., p. 308 (1901).

‡ "Ann. der Physik," Vol. XLII., p. 569 (1913).

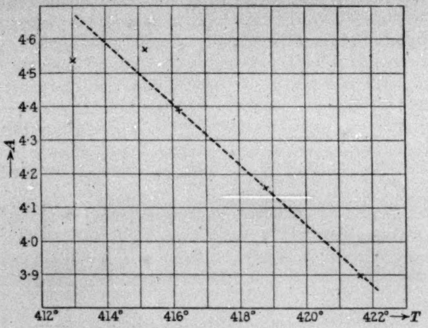

Diagram I.—Variation of the Clausius-Mosotti Expression (A) for Water-vapour as a Function of the Temperature.

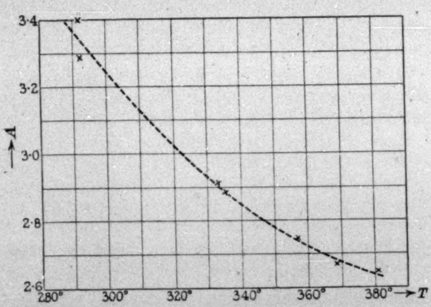

Diagram II.—Variation of the Clausius-Mosotti Expression (A) for Ammonia-vapour as a Function of the Temperature.

to 0·012 for a temperature decrease of 90° only. Yet, since we have no means of ascertaining from the present data the accurate shape of the curve in the case of water-vapour, we shall admit our straight-line extrapolation, without forgetting that the values of the dielectric constant thus obtained are almost certainly too low.

Let us now calculate the refraction undergone by electric waves in the atmosphere. Consider above the earth a layer of air having the height h. In order that the waves may follow exactly the curvature of the earth, the front ab, which is normal to its surface, must be displaced so that cd be also perpendicular at the point d (*see* Fig. 1).

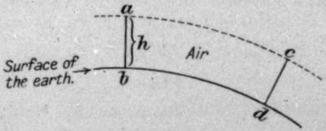

FIG. 1.

If we put r=the radius of the earth (mean= 6366·2 km.), this condition will be satisfied if the relative velocities $\frac{bd}{ac}$ are as $\frac{r}{r+h}$ or, as Kiebitz puts it, as $\frac{40,000}{40,000+2\pi}$.

We must now assume certain definite and regular conditions of the atmosphere—as far as possible the mean conditions that prevail. We suppose, as stated before, that the sending and receiving stations are separated by an ocean and that at sea-level (h_0), the pressure is 760 mm. At h_0 we can admit successively different temperatures according to the latitude. For the change of temperature with increasing h it is somewhat difficult to take an average value. I propose to consider a rather small positive temperature gradient for the first kilometer, say a decrease of 0·2 deg. for every 100 metres, on account of the fact that there is sometimes a negative gradient; in the second and third kilometre I propose to consider larger values of the gradient, say 0·5 deg. and 0·55 deg. respectively.

The relative humidity of air above the oceans will be put at 80 per cent.* For example, at 29·5° 1 kg. of moist air contains 19·53 gr. of water-vapour, when completely saturated; on our hypothesis it will contain only 15·62 gr. (or 0·01562 kg.). From the known relation $0{\cdot}623e/760-0{\cdot}377e=0{\cdot}01562$ we deduce the value of the vapour pressure e as 18·82 mm.

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* Hann, *loc. cit.*, p. 234.

For other values of *h*, the water vapour pressure diminishes above the oceans according to Suering's empirical formula * : $e_h = e_0\,10^{\frac{h}{6}\left(1+\frac{h}{20}\right)}$. In the case just considered (temp. $298^\circ_{abs.}$ at $h=0$) this would give for $h=1$ km., the value $e_h=$ 12·52 mm., corresponding to 11·58 gr. water-vapour in 1kg. of air. At this height the temperature would be 296°·0 (temperature gradient 0·2°) and the total pressure *P*, calculated from the well-known formula : $\log 760/P = h/18{,}400(1+at)\,(1+0{\cdot}377\,e/P)$ would be 677·9 mm. Hg. Under such conditions of temperature, moisture and pressure the density of air is 0·001173 at 298° and 0·001056 at 296°. For other values of *h* we get the figures indicated in Table II.

TABLE II.

h	$T^\circ k$	P	e	Specific moisture.	Density (moist air).	A (H_2O vapour).	k (moist air).
	Deg.						
0	298·0	760·0	18·82	15·62	0·001173	16·00	1·001407
0·5	297·0	719·9	15·47	13·53	0·001114	16·09	1·001230
1·0	296·0	677·9	12·52	11·58	0·001056	16·18	1·001070
2·0	292·5	603·5	8·11	8·43	0·000954	16·50	1·000830
3·0	287·0	536·3	5·01	5·84	0·000863	17·00	1·000650

Let us now consider the dielectric constant. For dry air we have $k=1{\cdot}000590$ † at 298° and 760 mm. For other temperatures and pressures within the range we have to consider, it has been shown that *A* remains absolutely constant. Thus the value of *A* for air (dry) will be put 0·152 in all cases. By means of the formula for binary mixtures,‡ we can calculate the value of the dielectric constant of air with 80 per cent. moisture at 298° and 760 mm. We get

$$\frac{k-1}{k+2}\cdot\frac{1}{0{\cdot}001173}=0{\cdot}152\times0{\cdot}98438+16{\cdot}0\times0{\cdot}01562.$$

From this we find $k=1{\cdot}001407$. If we calculate in the same way for other values of *h*, we get the figures of the last column of Table II. As just said, we may use a constant value for air (0·152) ; but for water-vapour we must take the values from the extrapolated curve, which are to be found in column 7.

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* "Wissensch Luftschiffahrten," Vol. III., p. 157 (1900).

† Boltzmann, "Pogg. Ann.," Vol. CLV., p. 403 (1873), and others.

‡ *See* Graetz, "Handb. der Elektr. u. des Magn.," Vol. I. (2), p. 175 (formula 81).

We have now to calculate, taking $k=1{\cdot}001407$ for the atmosphere at sea-level, what must be the values for different h's in order that the whole of the electromagnetic waves may be refracted along a curve corresponding to that of the earth's surface. Admitting that k is the square of the electric refractive index, we have that the velocity of propagation expressed in km. is equal to the velocity in vacuo divided by $\sqrt{k}$, thus $\frac{3{\cdot}10^{-5}}{\sqrt{(1{\cdot}001407)}}$ or $\frac{3{\cdot}10^{-5}}{1{\cdot}0007035}$ for $h=0$.

For any other value of h we have

$$\frac{6366{\cdot}2}{6366{\cdot}2+h}=\frac{\frac{3{\cdot}10^{-5}}{1{\cdot}0007035}}{\frac{3{\cdot}10^{-5}}{\sqrt{k_h}}}=\frac{\sqrt{k_h}}{1{\cdot}0007035}.$$

If we calculate $\sqrt{k_h}$ for the values of h considered and take the square, we get the figures under k' in Table III.

TABLE III.

h	k	k'	$k'-k$
0	1·001407	[1·001407]	...
0·5	1·001230	1·001250	+0·000020
1·0	1·001070	1·001092	+0·000022
2·0	1·000830	1·000778	−0·000052
3·0	1·000650	1·000464	−0·000186

We see that the figures under k, representing the actual dielectric constant, are first lower then higher than the figures under k'. If we put the difference between k and k' as ordinates and h as abscissa, the maximum of the curve will correspond to the height below which all waves are refracted towards the earth. This can be done accurately by calculating $\frac{dk}{dh}$, which is found to be equal to $\frac{dk'}{dh}$ $(=31{\cdot}4\times10^{-4})$, for $h=0{\cdot}825$ km.

The calculation can be repeated for other initial temperatures; *e.g.*, if we start from $T=296°$, we get the values of Table IV. The thickness of the layer in which the refraction is such as to give waves directed towards the surface of the earth is only about 350 metres under these conditions. Thus we rapidly reach a temperature for which only the air layer directly above the water refracts the whole amount of the electromagnetic

TABLE IV.

h	*T°k*	*P*	*e*	Specific moisture.	*d* moist air.	*A* water vapour.	*k* moist air.	*k′*
0	296·0	760·0	16·68	13·79	0·001183	16·18	1·001325	[1·001325]
0·5	295·0	717·6	13·76	12·05	0·0011122	16·27	1·001166	1·001167
1·0	294·0	677·3	11·09	10·29	0·001063	16·36	1·001017	1·001010
2·0	290·5	602·7	7·19	7·50	0·000956	16·67	1·000791	1·000675

waves towards the earth. Assuming that the results calculated would exactly correspond to the reality, we could state that in the tropical and subtropical regions, where the temperature of the atmosphere close to the water remains above 292°-293°, the conditions of the lower layers are such that the totality of the waves are refracted downwards. Thus we come to the conclusion that the electric "space" waves, instead of being lost for the major part before they reach a distant receiving antenna—as it was thought by Kiebitz—can, under favourable conditions, be totally refracted downwards; and under less favourable temperature conditions the greater part of them still reaches the receiver.

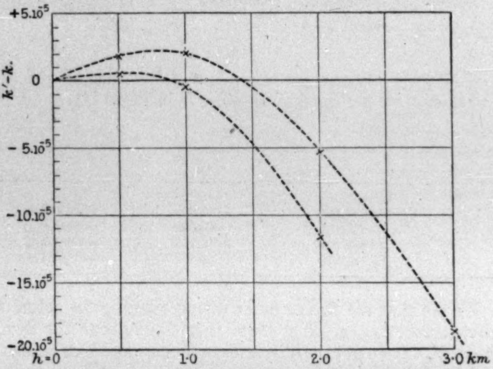

DIAGRAM III.—DIFFERENCE *K′−K* AT DIFFERENT HEIGHTS (*h* IN KILOMETRE) AND FOR DIFFERENT TEMPERATURE CONDITIONS.

But the actual figures arrived at can only be approximate, for (*a*) they depend upon the rather uncertain value of *k* for water-vapour; (*b*) they refer to ideal conditions, without regard to local changes (temperature, wind, clouds, rain, &c.). With regard to (*a*) it may again be emphasised that by employing a larger *k*-value for water-vapour—which would very likely better correspond to reality—the area above the oceans in which electric waves are totally refracted would extend to the temperate, and possibly even to arctic regions. With regard to (*b*) the discussion of local conditions would lead us to rather vague results as long as the point (*a*) is not settled; at any rate, it would only be worth while discussing the influence of local factors in connection with actual experimental data, which are lacking at present.*

Yet certain general conclusions can already be drawn; *e.g.*, since the average temperature above the oceans varies by less than 1° within 24 hours, the changes of intensity observed at the receiver, when comparing day with night, can hardly be the result of changes of the temperature and corresponding variations of the hygrometric state of the atmosphere.

Similar calculations could be made for the propagation of electric waves above the land. In this case the prismatic effect of the atmosphere is less important, since the relative amount of "space" waves is smaller (*see* Sommerfeld, *loc. cit.*). We shall not attempt any general calculation, because the temperature and other physical conditions are far less constant than above the oceans.

In conclusion, I think that, next to the ionisation of air, the variation of its electric refractive index—due principally to its hygrometric state—constitutes an important factor in the propagation of electromagnetic "space" waves and the intensity at a receiving wireless station.

ABSTRACT.

The author discusses the probable influence of moisture in the atmosphere on the refraction of electromagnetic waves round the earth's surface. The conclusion of Kiebitz that the presence of moisture does not affect the dielectric constant by more than 10 per cent. is shown to be erroneous, being based on the assumption that the Clausius-Mossoti formula, $\frac{k-1}{k+2}\cdot\frac{1}{d}=$constant, holds when passing from the liquid to the gaseous state. Examples are quoted to show

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* A serious effort to set up regular and accurate measurements in connection with wireless telegraphy has been started recently by an international committee, known under the name "Télégraphie sans fil scientifique" (T.S.F.S.), which has its central seat in Brussels. It had hardly started its activity when the work was interrupted by the war. This Paper is part of the work I have done in that laboratory, with a grant from the "Institut International de Physique Solvay," which I gladly acknowledge.

that this law fails in many cases, especially where the dielectric constant is high in the liquid state.

In the absence of more accurate data for ordinary temperatures, the author prefers to assume a value for the dielectric constant of water vapour obtained by extrapolating the results obtained by Baedeker for higher temperatures. The extrapolated value is almost certainly too low.

From this result, and the average conditions of the atmosphere over the ocean with regard to temperature gradient, &c., deduced from meteorological data, it is shown that the lowest layers of the atmosphere (1,000 to 1,500 metres approximately in depth) refract electromagnetic waves towards the earth, so that the greater part of the space waves will reach the receiver, contrary to the conclusion of Kiebitz.

DISCUSSION.

Dr. C. Chree (communicated remarks): The author, presumably unaware of their existence, does not refer to the somewhat numerous upper air data which have been published in this country by the Meteorological Office. A study of these data as to the temperature gradients would I think, have proved useful. Inversions are not confined to the lowest layers, but in these layers they are exceptional, and do not suffice to reduce the average temperature gradient to such low values as the author has taken for the first and second kilometres. For these, 5 deg. or 6 deg. per kilometre would not have been too high, especially for tropical regions. If the empirical exponential formula for vapour pressure be assumed, the pressure at any given height varies directly with that at ground level, and so in temperate latitudes is much lower in winter than in summer. It thus seems rather a fundamental point whether wireless phenomena in temperate latitudes show a marked annual variation corresponding with that of vapour pressure at ground level.

The Author, replying to Dr. Chree, said: A temperature decrease of 5 deg. to 6 deg. in the first kilometre may be or may not be more accurate than the decrease of 2 deg. which I have proposed; the scanty available observations above the oceans only allow us to state that both figures are probable values for the yearly average of the temperature gradient. If there were serious reasons to prefer the figures proposed by Dr. C. Chree, it would simply mean that the height of the air layer within which total refraction occurs is smaller than it appears from any calculations; but the order of magnitude of the phenomenon remains the same. The question of an inversion of the temperature gradient in upper atmospheric layers has not been considered in this connection. For large h's the waves are bent apart from the surface of the earth (see my calculations), and even a strong negative gradient is not likely to determine the bend in the other direction. It would be very interesting to verify how far a relation between vapour pressure and annual variation at a wireless receiver—as well as many other relations—is verified by actual observations. But, as far as the wireless phenomena are concerned, there are practically no data available at present. At the eve of starting such measurements it was precisely interesting to try to find out which could possibly be the meteorological factors most likely to affect the propagation of electromagnetic waves; and whatever may be disputed now or later with regard to the actual numbers given in my Paper, I think it will remain true that the amount of water vapour present in the atmosphere is one of the important factors.

IX. *Note on the Calculation of the Coefficient of Diffusion of a Salt at a Definite Concentration.* *By* A. Griffiths, *D.Sc.*

Received October 20, 1916. Revised January 26, 1917.

In the calculation of the coefficient of diffusion, by B. W. Clack, a simple relation is assumed between the density of a solution of a salt and the concentration. This simple relation is only approximately correct and compromises are made which require justification.

This note suggests—

(1) A method of calculating the coefficient of diffusion which, to a high degree of theoretical accuracy, gives values for the coefficient which are independent of a precise relationship between density and concentration; and

(2) Justifies the method of calculation adopted by B. W. Clack.

§ 1. Consider the case of a tube of unit cross-section and of length L_N. The lower end dips into a closed vessel which contains a solution whose concentration is N grams per cubic centimetre. The upper end opens into a large unclosed vessel containing water.

Let v_0=velocity of water at upper end of tube.
n=concentration at distance l from top of tube.
ρ_n=density of solution at distance l from top of tube.
c_N=mass of salt transmitted per second after attainment of the steady state when concentration at bottom of tube is N.
$f(n)$=coefficient of diffusion at concentration n.
δ_N=ratio of increment of volume of solution to mass of salt added at concentration N.

Then

$$\frac{v_0 n}{\rho_n - n} + f(n)\frac{dn}{dl} = c_N$$

$$v_0 = c_N \delta_N,$$

whence

$$\int_0^N \frac{f(n)(\rho_n - n)}{\rho_n - n + n\delta_N} dn = \int_0^{L_N} c_N dl = C_N L_N.$$

Differentiating both sides of the equation with respect to N

$$f(N)\left\{\frac{\rho_N - N}{\rho_N + N + N\delta_N} - \frac{d\delta'_N}{dN}\int_0^N \frac{f(n)}{f(N)} \frac{n(\rho_n - n)}{(\rho_n - n + n\delta_N)^2} dn\right\}$$

$$= \frac{d}{dN}(L_N C_N) \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The second term within the brackets ({ }) is relatively small; it is approximately correct to neglect it and to write

$$f(N)=\frac{d(L_NC_N)}{dN}\times\frac{(\rho_N-N+N\delta_N)}{(\rho_N-N)} \quad . \quad . \quad . \quad (2)$$

Thus, if the relation between L_NC_N and N be determined experimentally, the approximate value of the coefficient of diffusion at concentration N is given without any hypothesis as to the precise relationship between the density and the concentration.

Having found from equation (2) a first approximation to $f(N)$, and therefore to $f(n)$, (for the forms of the two functions are identical), the value of the second term within the brackets ({ }) can be determined; and, finally, equation (1) can be used to determine $f(N)$ with an increased and high degree of accuracy.

Let us consider the case of a solution which has the properties of an aqueous solution of KCl, but in which the relationship between the density and the concentration may be specified by

$$\rho_N=1+0{\cdot}62N-0{\cdot}13N^2.$$

This equation is approximately true for a solution of potassium chloride at 18°C. In the published work of B. W. Clack and of the author a linear relationship is assumed.

This equation gives

$$\delta_N=\frac{0{\cdot}38+0{\cdot}26N}{1+0{\cdot}13N^2}$$

and

$$\frac{d\delta_N}{dN}=\frac{0{\cdot}26-0{\cdot}0988N-0{\cdot}0338N^2}{(1+0{\cdot}13N^2)^2}.$$

The values of $f(N)$ deduced by intrapolation and extrapolation from B. W. Clack's Paper are as follows:—

$$f(0{\cdot}05)=1{\cdot}545\times10^{-5}\,;\quad f(0{\cdot}1)=1{\cdot}651\times10^{-5}$$
$$f(0{\cdot}15)=1{\cdot}757\times10^{-5}\,;\quad f(0{\cdot}20)=1{\cdot}863\times10^{-5}$$

The value of the integral in the second term within the brackets ({ }) may be found by plotting a curve on squared paper and when $N=0{\cdot}2$ (1) becomes, after a little re-arrangement,

$$f(N)\frac{\rho_N-N}{\rho_N-N+N\delta_N}\left\{1-0{\cdot}0050\right\}=\frac{dL_NC_N}{dN} \quad . \quad . \quad . \quad (3)$$

[Note.—$N=0{\cdot}2$.]

It may be particularly noted that equation (3) holds only for the special case that $N=0{\cdot}2$. A different value of N would necessitate a calculation of the new value of the second term within the brackets ({ }).

It may incidentally be mentioned that an approximate value of the integral within the brackets ({ }) of equation (1) is $N^2/2$. If this approximation be used then the second term within the brackets ({ }) of equation (2) becomes 0·0052.

§ 2. B. W. Clack in his Papers, writes :—

$$C_N L_N = {}_0K_N \left(1 - \frac{N\delta_N}{2}\right) N.$$

$$\therefore \frac{dC_N L_N}{dN} = N\left(1 - \frac{N\delta_N}{2}\right)\frac{d}{dN}({}_0K_N) + {}_0K_N\left(1 - N\delta_N - \frac{N^2}{2}\frac{d\delta_N}{dN}\right)$$

and, when $N = 0{\cdot}2$, from equation (3),

$$f(N) = \frac{(\rho_N - N + N\delta_N)\left[{}_0K_N\left(1 - N\delta_N - \frac{N^2}{2}\frac{d\delta_N}{dN}\right) + N\left(1 - \frac{N\delta_N}{2}\right)\frac{d}{dN}({}_0K_N)\right]}{(\rho_N - N)\{1 - 0{\cdot}0050\}}$$

$$= 0{\cdot}9998\,{}_0K_N + 0{\cdot}2104\frac{d}{dN}({}_0K_N) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

[Note.—$N = 0{\cdot}2$.]

The equation adopted by B. W. Clack in his recent Paper is

$$f(N) = {}_0K_N + \frac{\left(1 - \frac{N\delta_N}{2}\right)N}{1 - N\rho_N}\,\frac{d\,{}_0K_N}{dN},$$

whence

$$f(N) = {}_0K_N + 0{\cdot}2093\frac{d}{dN}({}_0K_N). \quad . \; . \; . \; . \; . \; . \quad (5)$$

[Note.—$N = 0{\cdot}2$.]

To compare the numerical values given by (4) and (5), let ${}_0K_N = 1{\cdot}7 \times 10^{-5}$, when $N = 0{\cdot}2$. Also let $\frac{d}{dN}({}_0K_N) = 0{\cdot}5 \times 10^{-5}$.

Equation (4) gives $f(N) = 1{\cdot}8049 \times 10^{-5}$; equation (5) gives $f(N) = 1{\cdot}8047 \times 10^{-5}$.

Thus, the equation used by B. W. Clack is completely justified.

It may be mentioned that the simple equation given by B. W. Clack in an earlier Paper $\left(\text{viz., } f(N) = {}_0K_N + N\frac{d}{dN}({}_0K_N)\right)$ gives a good approximation to the correct value.

ABSTRACT.

In the calculation of the coefficient of diffusion, by B. W. Clack, a simple relation is assumed between the density of a solution of a salt and the concentration. This simple relation is only approximately correct, and compromises are made which require justification.

This note—

1. Suggests a method of calculating the coefficient of diffusion which, to a high degree of theoretical accuracy, gives values for the coefficient which are independent of a precise relationship between density and concentration; and
2. Justifies the method of calculation adopted by B. W. Clack.

DISCUSSION.

Mr. B. W. Clack: The coefficient of diffusion is not a constant, but varies with the concentration to a considerable extent. In most experimental work the coefficient varies at different positions in the apparatus, and any work which helps towards the proper interpretation of results is of value.

X. *A Special Test on the Temperature Effect of Gravitation.**

By P. E. Shaw, *B.A., D.Sc., and* C. Hayes, *B.Sc.*

Received December 29, 1916

1. *Introduction.*

In the "Philosophical Transactions" of the Royal Society for 1916 † there is a Paper by one of us dealing with the possible existence of a temperature coefficient in the law of gravitation, and giving an account of experiments made to discover this coefficient.

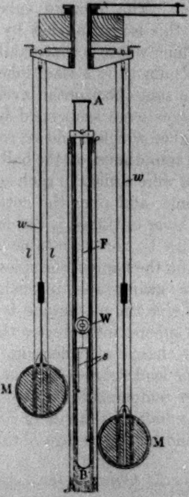

Fig. 1.—An Elevation of the Apparatus.

The central vacuum tube contains the torsion balance whose movements are observed through the window W. The special brass cylinders on which the contacts are made are shown thick on the hanging wires *w w*.

The essential parts of the apparatus used are shown in Fig. 1.

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* Briefly described in the discussion on Gravitation at a meeting of Section A, British Association, Newcastle, 1916.

† Shaw, Vol. CCXVI., pp. 349-392.

From this sketch many details have been omitted, but full drawings and description will be found in the Paper quoted.

The glass tube A B is 120 cm. long and 5 cm. bore, and is exhausted to a pressure of 15 mm. of mercury. Two fine wires, *ss*, of length 15 cm. and 35 cm. respectively, are suspended from the ends of a torsion beam which is hung by a quartz fibre, F, 60 cm. in length. Each of these wires carries a very pure silver ball weighing 2·5 gm. The movements of the torsion beam, carrying a mirror which reflects light from a distant scale, are observed through the window W by a telescope 4·5 metres distant. The vacuum tube is well lagged with cotton wool, and this is surrounded by a helix of "compo" tubing through which water is continually passing.

Outside the vacuum tube a lead sphere, weighing 47 kilos, was placed on the same horizontal level as each of the silver balls. These spheres were suspended by stout copper wires, *w w*, from a turntable which could be rotated about a vertical axis with ease and smoothness on the ball-bearings shown in the figure Two holes were drilled in each sphere to admit a heating coil of nichrome, and over the cotton-wool covering the spheres there is a layer of tinfoil. The leads *l l* for the heating coils are carried up to the turn-table arms, and, therefore, exert no influence on the hang of the spheres. Stops are placed on the dial of the swing-table in such positions (found by experiment) as to give the maximum deflection of the torsion system as the lead spheres moved from the A to the B positions (*see* Fig. 2). This change of deflection, called the range, was measured when the lead spheres were at 18°C., and also when they were at other temperatures up to 250°C. The temperature coefficient was found by dividing the change of range by the whole range and by the change of temperature.

2. *Object of the Test.*

From these experiments it was deduced that, at least for the materials used, there is a temperature effect of gravitation when the temperature of the large mass rises, while that of the small mass remains constant.

Supposing that this effect is linear it can be expressed by the relation

$$F=G(1+a\theta)\frac{Mm}{d^2},$$

where F is the force of attraction and θ is the increment in temperature of the large mass, G being the Newtonian constant at ordinary temperatures, and d being the distance between the centre of mass of either large sphere and its corresponding small mass. The value found for a was $+1{\cdot}2\times10^{-5}$ per 1°C.

There are many possible sources of error mentioned in the Paper which might account for the observed small change in the deflection. All these were considered to have been satisfactorily eliminated by the precautions taken; but another possible source was mentioned in the criticism after the reading of the Paper. This will be found on p. 391 as follows: "It is suggested that with the high temperature of the lead spheres (250°C.) considerable convection currents would be set up

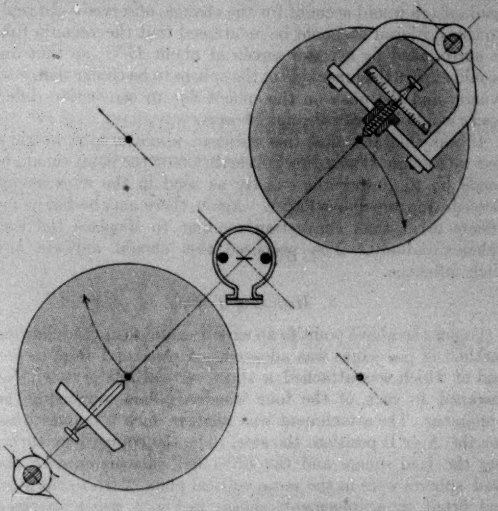

FIG. 2.—A PLAN VIEW.

The large spheres are shown with the hanging wires in section at their centres. The section of the vacuum tube is seen at the centre. The micrometer and frame are shown on the right and the micrometer alone on the left for the A position. The B position is indicated by a section of the hanging wires.

round them even when lagged.'' In the region where the spheres are close to the tube, the air velocity might be very much larger than on the outer regions; and, as a consequence, a difference of pressure would be called into play, pushing the suspended spheres towards the tube, and therefore increasing the actual attraction on the suspended small balls. It is calculated that an *inward* displacement of 0·15 mm. in each lead sphere would account for the change observed. In reply to this criticism it should be mentioned that the vacuum tube is surrounded by a water screen at about 11°C., so that one would expect the inner side of the sphere to be colder than elsewhere, and the push on the sphere due to convection differences would be *outwards*, not *inwards*.

It was thus felt that this criticism would not be weighty, but we thought that a direct observation on the point should be made on the apparatus exactly as used in the experiments described in the above Paper. Again, there may be disturbing effects other than convection tending to displace the lead spheres radially. The proposed test should indicate any such influence.

3. *Method of Testing.*

To put the above point to an experimental test, the following method of procedure was adopted. A piece of T-iron, to one end of which was attached a short vertical rod of steel, was fastened to each of the four wooden pillars supporting the apparatus. The attachment was made in such a manner that for the A or B position, the steel rods, the copper wire carrying the lead sphere and the silver ball (associated with the lead sphere) were in the same vertical plane. On to the steel rod fitted an arrangement, shown in Fig. 2, which carries a micrometer. In use this micrometer screw was adjusted by swinging the micrometer on its vertical pin and by turning the screw until it just made contact with the vertical wire. When this occurred an electric circuit was completed through the wire and the micrometer, which caused a sound in a telephone receiver, and the reading of the micrometer was then taken. The position of the copper wire was thus accurately determined, and any movement to or from the silver sphere could be detected. Unless the copper wire were quite straight, we should not be able to repeat our readings if any rotation occurred about a vertical axis passing through the wire.

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Feb. 13, 1903.	Righi, Prof. Aug. *The University, Bologna.*
Feb. 12, 1892.	Van der Waals, Professor J. D. *The University, Amsterdam.*

Ex-Officio Fellows.

Dec. 20, 1910.	The President of the American Physical Society.
Mar. 16, 1878.	The President of the French Physical Society.
Feb. 14, 1902.	The President of the German Physical Society.

Ordinary Fellows.

The names of Original Members are marked thus **; the names of those who were *admitted* Members during the first Session of the Society are marked *; Life Fellows are marked ‖; Members who have served on the Council are marked §; and Members who have served as Vice-Presidents are marked †.

The letters F.P.S.L. have been formally adopted as the indication of Fellowship of the Physical Society of London.

Jan. 15, 1876.	ABNEY, Sir W. de W., K.C.B., F.R.S. (PAST PRESIDENT). *Rathmore Lodge, Bolton Gardens South,* S.W.
Mar. 10, 1911.	Abraham, R. M. *Gurney Street Works, Walworth,* S.E.
May 5, 1914.	Abram, Harold H. *Physics Dept., Guy's Hospital, London Bridge,* S.E.
June 28, 1912.	Adams, P. F. 81 *Rock Avenue, Gillingham, Kent.*
May 12, 1888.	Addenbrooke, G. L. 35 *Holland Villas Road,* S.W.
Nov. 24, 1911.	Airey, H. Morris. 24 *Taswell Road, Southsea, Hants.*
Apr. 28, 1911.	Airey, J. R., M.A., B.Sc. 73 *Claremont Road, Forest Gate,* E.
Jan. 26, 1906.	ALLEN, H. Stanley, M.A., D.Sc. (MEMBER OF COUNCIL). 5 *Presburg Road, Malden, Surrey.*
Mar. 10, 1911.	Allen, W. H., M.Inst.C.E. *Bromham House, Bromham, nr. Bedford.*

Date of Election.	
Mar. 10, 1916.	Amor, A. H., M.E. 2 *Idrone Terrace, Blackrock, Co. Dublin*
Dec. 4, 1891.	‖Anderson, Prof. Alex., M.A., LL.D. *Queen's College, Galway.*
Nov. 9, 1894.	Appleyard, Rollo, M.Inst.C.E. 79 *St. Mary's Mansions, Paddington*, W.
Mar. 22, 1895.	§‖Armstrong, Prof. H. E., Ph.D., LL.D., F.R.S. 55 *Granville Park, Lewisham*, S.E.
Jan. 22, 1897.	Ashworth, J. R., D.Sc. 55 *King Street South, Rochdale.*
May 24, 1907.	Ayrton, Mrs. Hertha. 41 *Norfolk Square*, W.
Nov. 12, 1887.	Bacon, Vice-Admiral R. H. S., R.N. *Fleet House, Dover.*
Mar. 27, 1903.	Baillie, G. H. 82 *Victoria Street*, S.W.
Nov. 24, 1911.	Baker, Bevan B., B.Sc. *Frontenac, Donnington Road, Harlesden*, N.W.
Jan. 26, 1912.	Barker, S. G., Ph.D. *Maharajah's College, Trivandrum, S. India.*
Feb. 25, 1910.	§Barkla, Prof. C. G., D.Sc., F.R.S. 34 *Priestfield Road, Edinburgh.*
May 22, 1914.	Barnard, J. E. *Park View, Brondesbury Park*, N.W.
June 9, 1899.	Barnes, E. *C/o Priest & Ashmore, Newton Works, Eyre Street, nr. Moorhead, Sheffield.*
Feb. 28, 1902.	Barr, J. M. McG., *The Savile Club, Piccadilly*, W.
Feb. 25, 1910.	Barratt, T., D.Sc. *Wandsworth Technical Institute, Wandsworth*, S.W.
	§**‖Barrett, Prof. Sir W. F., F.R.S. 31 *Devonshire Place*, W.
Mar. 11, 1892.	BARTON, Prof. Edwin Henry, D.Sc., F.R.S., F.R.S.E., A.M.I.E.E. (MEMBER OF COUNCIL). *University College, Nottingham.*
Jan. 21, 1898.	Bartrum, C. O., B.Sc. 32 *Willoughby Road, Hampstead*, N.W.
	§*‖Baynes, R. E., M.A. *Christ Church*, and 2 *Norham Gardens, Oxford.*
Nov. 28, 1913.	Beach, A. C. G. 49 *Park Road, West Dulwich*, S.E.
Feb. 28, 1913.	Beattie, Prof. J. C. *South African College, Cape Town, S. Africa.*
May 22, 1886.	Bell, Chichester A., M.B. "*Torbrex*," *Headington Hill, Oxford.*
May 22, 1914.	Benke, L. L. *Sunny Mount, Overhill Road, Dulwich*, S.E.
Nov. 27, 1903.	Bennett, Wm., B.Sc. *School House, Brighton College, Brighton.*
Nov. 25, 1910.	Berry, G. H. 3 *Station Road, Winchmore Hill*, N.
Apr. 26, 1907.	Blackie, Alfred, Junr. 24 *Alexander Square, S. Kensington.*
Mar. 12, 1909.	Blaikie, L., M.A. *Civil Service Commission, Burlington Gardens*, W.
May 28, 1881.	‖Blaikley, D. J. 13 *Elsworthy Terrace, Primrose Hill*, N.W.
Nov. 28, 1885.	§†Blakesley, T. H., M.A., M.Inst.C.E. 7 *Cromwell Crescent, Earl's Court, Kensington*, W.
Mar. 8, 1907.	Boardman, E., B.Sc., F.C.S. 3 *Glade Street, Bolton.*
Mar. 8, 1895.	Borns, H., Ph.D. 5 *Sutton Court Road, Chiswick*, W.
Mar. 8, 1901.	Bose, Prof. Jagadis Chunder, M.A., D.Sc. *C/o Messrs. H. S. King & Co.*, 9 *Pall Mall*, S.W.

Date of Election.	
Feb. 13, 1914.	Boulding, R. S. H., B.Sc. *"Woodthorpe," Barston Road, W. Norwood.*
	*‖Bourne, C. W., M.A. *Staplehurst Rectory, Kent.*
Feb. 12, 1892.	‖Bower, William Richard. *Technical College, Huddersfield.*
June 12, 1880.	‖BOYS, Prof. C. Vernon, F.R.S. (PRESIDENT). 66 *Victoria Street*, S.W.
Feb. 22, 1907.	§Bragg, Prof. W. H., M.A., F.R.S. *University College, Gower Street*, W.C.
Nov. 10, 1916.	Briggs, Lt. R. A., R.E. *Cromer Villa, Shakespeare Street, Nottingham.*
Mar. 10, 1905.	Brinkworth, J. H., B.Sc. 31 *Albert Mansions, Battersea Park*, S.W.
Nov. 21, 1874.	‖Brown, Prof. F. D., M.A. *University College, Auckland, N.Z.*
Nov. 22, 1912.	Brown, F. G. W., B.Sc., F.C.P. *Strathmore*, 25 *Wellwood Road, Goodmayes, Essex.*
Dec. 14, 1894.	‖Bryan, Prof. G. H., Sc.D., F.R.S. *Plas Gwyn, Bangor.*
Apr. 14, 1916.	Bryan, G. B., D.Sc. *Royal Naval College, Greenwich.*
June 11, 1915.	Brydon, Sydney. 7 *Wilkie Street, Newcastle-on-Tyne.*
Feb. 25, 1882.	‖Buchanan, J., D.Sc. *Gordon's College, Aberdeen.*
May 10, 1895.	‖Buckmaster, C. A., M.A. 16 *Heathfield Road, Mill Hill, Park, Acton*, W.
Nov. 26, 1909.	Bullerwell, J. W., M.Sc. 57 *Hunters Road, Newcastle-on-Tyne.*
Feb 11, 1916.	Burbidge, P., M.Sc. *Victoria University College, Wellington, New Zealand.*
Nov. 23, 1894.	Burke, John Butler, M.A. 58 *Belsize Park Gardens*, N.W.
Mar. 25, 1904.	‖Burns, G. J., B.Sc. 209 *Shooters Hill Road, Blackheath*, S.E.
Jan. 25, 1907.	Burrow, S. G., B.Sc. *Baroda College, Baroda, India.*
Dec. 9, 1898.	‖CALLENDAR, Prof. H. L., M.A., LL.D., F.R.S. (PAST PRESIDENT). *Imperial College of Science, South Kensington*, S.W.; and 49 *Grange Road, Ealing*, W.
Apr. 26, 1895.	§†Campbell, A., B.A. *Lalbagh, Park Road, Teddington, Middlesex.*
Nov. 24, 1916.	Campbell, Col. J. C. L. *Achalader, Blairgowrie.*
Dec. 14, 1894.	‖Capstick, J. W., M.A., D.Sc. *Trinity College, Cambridge.*
Nov. 24, 1916.	Cecil, Lt. the Hon. Henry Mitford Amhurst, R.N. *H.M.S. "Inflexible," c/o G.P.O.*
Apr. 10, 1908.	§Chalmers, S. D. M.A. 13 *Ribblesdale Road, Hornsey, N.*
Apr. 26, 1884.	‖Chattock, A. P., D.Sc. *Heathfield Cottage, Crowcombe, near Taunton.*
Dec. 11, 1896.	CHREE, C., M.A., Sc.D., LL.D., F.R.S. (PAST PRESIDENT). 75 *Church Road, Richmond, Surrey.*
June 20, 1914.	Christie, Capt. John, *Eton College, Windsor.*
May 12, 1877.	§‖Christie, Sir W. H. M., K.C.B., M.A., F.R.S. *Tower House, Down, Kent.*
Mar. 8, 1912.	Churton, Thos. Harding, M.I.E.E. *Allerton Hill, Chapel Allerton, Leeds.*
Dec. 9, 1904.	Clack, B. W., M.Sc. *Birkbeck College, Bream's Buildings, Fetter Lane*, E.C.

Date of Election.	
Nov. 22, 1895.	Clark, G. M., M.A. *C/o Victoria Falls & Transvaal Co. P.O. Box* 2671, *Johannesburg, S. Africa.*
May 8, 1914.	Clarke, E. Russell. 11 *King's Bench Walk, Temple,* E.C.
June 9, 1911.	Clarke, Capt. F. C. 37 *Conyers Road, Streatham,* S.W.
Dec. 10, 1881.	‖Clayden, A. W., M.A. 5 *The Crescent, Mount Radford, Exeter.*
Nov. 27, 1875.	‖CLIFTON, Prof. R. B., M.A., F.R.S. (PAST PRESIDENT). 3 *Bardwell Road, Oxford.*
Nov. 24, 1899.	Clough, W. T. "*Chelladene,*" *St. Margarets Road, Wanstead Park, Essex.*
Mar. 23, 1889.	‖Cockburn, A. C., F.C.S. 156 *Willesden Lane,* N.W.
Mar. 29, 1903.	Codd, M.A. 32 *Half Moon Lane, Herne Hill,* S.E.
Mar. 12, 1909	Coker Prof. E. G., M.A., D.Sc., F.R.S. *University College, Gower Street,* W.C.
May 16, 1890.	‖Coleman, Prof. W. M., A.B. *C/o* J. M. Collier, Esq., *Pittsburg, Texas, U.S.A.,* and "*Woodview,*" *Bishopswood Road, Highgate, N.*
Nov. 24, 1911.	Cooper, Richard. "*Redcroft,*" *Carew Road, Northwood, M'x.*
Feb. 22, 1895.	COOPER, W. R., M.A., B.Sc. (VICE-PRESIDENT). 113 *Tulse Hill,* S.W.
Nov. 28, 1913.	Coursey, P. R., B.Sc. (Eng.). "*Gwynash,*" 3 *Coniston Road, Muswell Hill,* N.
Nov. 22, 1912.	Cox, Prof. John. 41 *Gordon Square, London,* W.C.
Mar. 8, 1895.	‖Crawley, C. W. S. 89 *Upper Richmond Road, Putney.*
	*‖Crawshaw, Edward. 25 *Tollington Park,* N.
	*‖Crisp, Sir Frank, Bart., B.A., LL.B., J.P. 5 *Lansdowne Road,* W.; and *Friar Park, Henley-on-Thames.*
Apr. 18, 1890.	§‖Croft, William Bleadon, M.A. 9 *College Street, Winchester.*
Mar. 23, 1889.	§‖Crompton, Lt.-Col. R. E. B., C.B. *Thriplands, Kensington Court,* W.
May 12, 1888.	§**‖Crookes, Sir W., F.R.S. 7 *Kensington Park Gardens,* W.
Mar. 13, 1886.	‖Cunynghame, Sir H. H., K.C.B. *Home Office, Whitehall,* S.W.
	‖Curtis-Hayward, A. C., M.A. *Maxwell House, Arundel Street, Strand,* W.C.
May 12, 1911.	Dakin, A., M.A., B.Sc. 97 *Park Grove, Hull.*
Mar. 8, 1912.	Darling, Chas. R. 186 *Eglinton Road, Plumstead,* S.E.
Mar. 26, 1898.	Darwin, Horace, M.A., F.R.S. *The Orchard, Cambridge.*
May 12, 1877.	‖Darwin, Major L., R.E., F.R.S. 12 *Egerton Place,* S.W.
Dec. 4, 1891.	‖Davey, Henry, M.Inst. C.E. *Conaways, Ewell, Surrey.*
Feb. 26, 1904.	Davidge, Prof. H. T., B.Sc., M.I.E.E. *Ordnance College, Woolwich,* S.E.
Mar. 11, 1898.	Davidson, Sir J. M., M.B., C.M. 26 *Park Crescent, Portland Place,* W.
Mar. 26, 1909.	Davies, B. *Wilford, Chase Green Avenue, Enfield.*
Dec. 18, 1914.	Davison, L. R. *Sharlston, Kingsway, Woking, Surrey.*
Feb. 13, 1875.	‖Day, R. Evan. *Culver, Plaistow Lane, Bromley, Kent.*
June 24, 1876.	Dewar, Sir James, F.R.S. *Cambridge;* and *Royal Institution, Albemarle Street,* W.
Nov. 23, 1900.	Dickinson, R. H. *Technical School, Glossop, nr. Manchester.*
Oct. 30, 1896.	§Donnan, Prof. F. G., M.A., Ph.D., F.R.S. *Chemical Laboratories, University College, Gower Street,* W.

Date of Election.	
Feb. 9, 1917.	Dowsett, H. M., M.I.E.E., M.I.R.E. *Middleton House, Victoria Road, Colchester.*
Mar. 11, 1898.	Drysdale, C. V., D.Sc. *Queen Anne's Chambers, Westminster*, S.W.
Mar. 2, 1900.	†DUDDELL, W., F.R.S. (Treasurer). 56 *Victoria Street*, S.W.
Jan. 23, 1914.	Duffield, Prof. W. G., D.Sc. 18 *Redlands Road, Reading.*
	*‖Durham, Henry. 13 *Coleherne Road, South Kensington*, S.W.
Feb. 14, 1902.	Dutton, F., B.Sc. "*Rivington*," *Woodwarde Road, Dulwich*, S.E.
May 28, 1915.	Dye, W. D. 13 *Merton Hall Road, Wimbledon.*
May 26, 1877.	‖Dyer, Sir W. T. Thiselton, K.C.M.G., M.A., B.Sc., F.R.S. *The Ferns, Witcombe, Gloucester.*
Nov. 28, 1902.	ECCLES, W., D.Sc. (Secretary). *Finsbury Technical College, Leonard Street*, E.C. 2; 26 *Ridgmount Gardens, Gower Street*, W.C.
Feb. 24, 1905.	Edgcumbe, K., M.Inst.C.E. *Aldenham Grange, Watford.*
Feb. 12, 1892.	‖Edser, Edwin. 3 *Hillyfields Crescent, Brockley*, S.E.
Feb. 24, 1911.	Egerton, A. C. G. 19 *Old Court Mansions*, W.
June 26, 1903.	Eggar, W. D., M.A. *Eton College, Windsor.*
Dec. 11, 1886.	§†Elder, H. M., M.A. *Albemarle House*, 28B *Albemarle Street*, W.
Nov. 28, 1913.	Ellis, E.R. 179 *Warwick Road, Kensington*, S.W.
June 26, 1891.	Enright, James, B.Sc. 19 *Sydenham Road* (*North*), *Croydon.*
Apr. 11, 1902.	Erskine-Murray, J. R., D.Sc., F.R.S.E. 4 *Great Winchester Street, London*, E.C.
May 27, 1904.	Etchells, E. F., A.M.I.M.E. *Building Acts' Office, L.C.C., Spring Gardens*, S.W.
Mar. 8, 1895.	Eumorfopoulos, N., B.Sc. 24 *Pembridge Gardens*, W.
Feb. 21, 1890.	‖Evershed, Sydney. *Acton-Lane Works, Chiswick*, W.
Apr. 22, 1898.	§†Ewing, Sir J. A., F.R.S. *Edinburgh University.*
June 9, 1911.	Farr, C. C. *Canterbury College, Christchurch, New Zealand.*
Nov. 28, 1902.	Finn, M., M.Sc. *Sea View, Ambleside Drive, Southend-on-Sea.*
Dec. 14, 1894.	‖Fitzpatrick, Rev. T. C., M.A. *The Lodge, Queen's College, Cambridge.*
	§†*Fleming, Prof. J. A., M.A., D.Sc., F.R.S. *University College, Gower Street*, W.C.
Mar. 25, 1882.	§†‖Fletcher, Sir Lazarus, M.A., F.R.S. *British Museum* (*Natural History*), *Cromwell Road*, S.W.
Apr. 14, 1883.	§‖Forbes, Prof. G., M.A., F.R.S. 11 *Little College Street, Westminster.*
Nov. 25, 1898.	Forsyth, R. W. *Imperial College of Science, London*, S.W.
Feb. 28, 1913.	Fortescue, Prof. C. L. *H.M.S. Vernon, Portsmouth.*
	**‖FOSTER, Prof. G. Carey, F.R.S. (Past President). *Ladywalk, Rickmansworth, Herts.*
June 14, 1879.	‖Fox-Pitt, St. George Lane. *Travellers' Club, Pall Mall*, S.W.
Jan. 21, 1898.	‖Francis, William. 11 *Liverpool Road, Kingston Hill, Surrey.*
Dec. 13, 1895.	‖Fridlander, E. D. *Hertford Place, Coventry.*
Nov. 25, 1892.	‖Fry, Francis James, J.P. *Messrs. Fry & Sons, Bristol.*

Date of Election.	
Feb. 22, 1895.	Gardiner, J. H. 59 *Wroughton Road, Balham*, S.W.
Feb. 22, 1907.	Garrett, C. A. B., B.Sc. 94 *Granville Road, Blackburn.*
Nov. 1, 1889.	Garrett, T. A., M.A. 23 *Wynnstay Gardens, Kensington*, W.
Nov. 13, 1914.	George, H. T., M.A., M.R.C.S., L.R.C.P. 33 *Ampthill Square, London*, N.W.
May 13, 1892.	‖Gerrans, H. T. 20 *St. John's Street, Oxford.*
Nov. 13, 1896.	Gifford, J. W., Lt.-Col. *Oaklands, Chard, Somerset.*
Dec. 13, 1879.	‖GLAZEBROOK, R. T., C.B., D.Sc., F.R.S. (PAST PRESIDENT). *Bushy House, Teddington, Middlesex.*
Nov. 28, 1907.	Gordon, Rev. J. M. 7 *Moreton Gardens, S. Kensington.*
Mar. 23, 1889.	‖Gorham, J. M., M.Inst. C.E. *Drove House, Singleton, Sussex*
Mar. 9, 1917.	Grace, A. W. "*Evandale,*" *Granville Road, Sidcup.*
Feb. 23, 1912.	Green, Harry H., B.Sc. 61 *Durham Road, Manor Park, Essex.*
Mar. 11, 1910.	Griffith, I. O., M.A. 49 *Banbury Road, Oxford.*
Feb. 22, 1896.	§Griffiths, A., D.Sc. *Birkbeck College, Bream's Buildings, Chancery Lane*, E.C.
	Griffiths, E. 4 *Meadow Cottages, Waldagrave Road, Teddington.*
Feb. 23, 1884.	‖Grundy, J. *Ruislip, Teignmouth Road, Cricklewood*, N.W.
Jan. 22, 1915.	Guild, J., A.R.C.Sc., D.I.C., F.R.A.S. *Lessaleigh*, 63 *Hampton Road, Teddington, Middlesex.*
Mar. 2, 1900.	‖Gundry, P. G. Ph.D., B.Sc. *Transvaal University College, Box* 43, *Pretoria, S. Africa.*
Mar. 12, 1909.	§Hadfield, Sir R. A., F.R.S. 22 *Carlton House Terrace*, S.W.
Apr. 14, 1916.	Hague, B. 51 *Queen's Road, South Farnborough, Hants.*
Mar. 13, 1908.	Hall, A. E. *Royal College of Science, South Kensington.*
Mar. 10, 1916.	Hall, W. J. 15 *Derby Street, Walsall.*
May 8, 1914.	Hallimond, A. F. *Museum of Practical Geology*, 28 *Jermyn Street*, S.W.
	Hare, A. T. *The Orangery, St. Margaret's, Middlesex.*
Feb. 14, 1880.	§Harker, J. A., D.Sc., F.R.S. *Alston, Queen's Road, Teddington, Middlesex.*
Mar. 12, 1909.	Harlow, F. J., B.Sc. *St. Nicholas, Castlebar Park Road, Ealing.*
Jan. 22, 1909.	Harris, T., B.Sc. 2 *Apsley Villas, Acton, London*, W.
Mar. 2, 1900.	Harrison, Prof. E. P. *The Observatory, Alipore, Calcutta.*
Apr. 28, 1893.	‖Harrison, Fred., M.A. *School House, Newcastle, Staffordshire.*
Jan. 24, 1896.	Harvey, H. S. S. 270 *G.P.O., Brisbane, Queensland.*
Jan. 13, 1893.	‖Haselfoot, C. E., M.A. *Hertford College, Oxford.*
Apr. 26, 1896.	Hay, Prof. A., D.Sc. *Imp. Inst. Science, Bangalore, Mysore, India.*
May 6, 1882.	‖Heaton, W. H., M.A. *University College, Nottingham.*
Nov. 26, 1909.	Hemmy, Prof. A. S. *Lahore. India.*
	§**Henrici, Prof. O., Ph.D., F.R.S. *Hiltingbury Lodge, Chandler's Ford, Hants.*
May 9, 1885.	‖Herroun, E. F. *Ravenshaw, Croydon Road, Reigate, Surrey.*
Mar. 16, 1878.	Hicks, Prof. W. M., M.A., F.R.S. *Leamhurst, Ivy Park Road, Sheffield.*
Dec. 11, 1875.	‖Higgins, Clement, M.A. *Winterdyne, Bournemouth West.*
Mar. 24, 1911.	Hirst, Hugo. 71 *Queen Victoria Street, London*, E.C.

Date of Election.	
Jan. 22, 1915.	Hitchins, A. B. *The Ansco Coy. Research Laboratory, Binghampton, N.Y.*
Nov. 28, 1913.	Hodson, F., B.Sc., Ph.D. *Bablake School, Coventry.*
Feb. 7, 1890.	‖Holland, Charles E., B.A. 4 *Redington Road, Hampstead,* N.W.
Nov. 22, 1901.	‖Hopkinson, Prof. B., F.R.S. 10 *Adams Road, Cambridge.*
Apr. 22, 1882.	‖Hopkinson, Edward, M.A., D.Sc. *Ferns, Alderley Edge, Cheshire.*
Apr. 26, 1907.	Hopwood, F. Lloyd, M.Sc., A.R.C.Sc. 416 *Alexandra Park Road, Alexandra Park,* N.
Jan. 24, 1884.	Howard, Alfred, A.R.S.M., A.R.C.Sc. 186 *Stanstead Road, Forest Hill,* S.E.
May 13, 1876.	‖Howard, David. *Devon House, Buckhurst Hill, Essex.*
Jan. 24, 1908.	Howarth, J. G. *Danes Dyke, Reigate Road, Reigate.*
Feb. 25, 1910.	HOWE, Prof. G. W. O., D.Sc. (Member of Council.) 22 *Dorset Road, Merton Park, Surrey.*
Feb. 8, 1907.	Hunter, J. de G., M.A. *Survey of India, Dehra Dun, United Provinces, India.*
	**‖Huntington, Prof. A. K. *King's College, Strand,* W.C.
Dec. 24, 1915.	Insull, Samuel. *Edison Buildings,* 72 *West Adams Street, Chicago, Ill., U.S.A.*
Mar. 25, 1882.	‖Jackson, M. J., M.A., D.Sc. *Applegarth, Aldergrove, British Columbia.*
Mar. 8, 1879.	‖Jacob, F. *Messrs. Siemens Bros. & Co., Woolwich,* S.E.
May 22, 1889.	‖Johnson, W. Claude. *Broadstone Farm, Coleman's Hatch, Sussex.*
Apr. 8, 1910.	Jolley, A. C. *Northampton Polytechnic Institute, Clerkenwell, London,* E.C.
Nov. 22, 1912.	Jolly, H. L. P. 69 *Teddington Park, Teddington, Middlesex.*
Feb. 17, 1886.	‖Jones, D. E., B.Sc. *Eryl-Daf, Radyr, Cardiff.*
Feb. 26, 1904.	Jones, Prof. E. Taylor, D.Sc. *University College, Bangor.*
Nov. 27, 1880.	‖Jones, H. Chapman, F.I.C. "*Birnam,*" 67 *Shakespeare Road, Hanwell,* W.
Nov. 27, 1908.	Jones, Prof. R. Ll. *The Observatory, Madras, India.*
Nov. 22, 1912.	Jordan, F. W., B.Sc. 42 *Evelyn Avenue, Newhaven, Sussex.*
Dec. 15, 1877.	‖Judson, J. E., B.A. *Priors, Godalming.*
Mar. 10, 1911.	Kaye, Capt. G. W. C., D.Sc. 76 *Addison Gardens, Kensington,* W.
June 26, 1914.	Kebby, C. H. 75, *Sterndale Road, West Kensington Park,* W.
Nov. 24, 1911.	Kempson, E. W. E. 2 *Whitehall Gardens, Rugby.*
May 16, 1913.	Kent, Henry Charles. 179 *City Road, Edgbaston, Birmingham.*
	*‖Kesselmeyer, C.A. *Rose Villa, Vale Road, Bowdon, Cheshire.*
	*‖Kesselmeyer, W. J. 19 *Broomfield Lane, Hale, nr. Altrincham, Cheshire.*
Jan. 23, 1914.	Khan, Mohammed A. R. *Nizam College, Hyderabad, Deccan, India.*
Dec. 5, 1874.	Kinch, Prof. Edward, F.I.C. *Komaba, Haslemere, Surrey.*
Feb. 24, 1905.	Kingsbury, J. E.. *Burton Dassett, Herne Hill,* S.E.

Date of Election.	
Apr. 14, 1905.	Kirkman, J. C., B.Sc. *Technical College, Mount Pleasant, Swansea.*
Mar. 2, 1878.	‖Kirkman, J. Parke, M.A. 64 *Kimbolton Road, Bedford.*
Mar. 13, 1896.	Korda, Désiré. 115 *Rue de Courcelles, Paris.*
	*‖Lambert, Prof. C. J., M.A. *Buckhurst Lodge, Westerham Hill, Kent.*
Jan. 25, 1879.	‖Lankester, Prof. Sir E. Ray, K.C.B., M.A., F.R.S. 29 *Thurloe Place*, S.W.
June 9, 1911.	Larard, C. E., M.Inst.C.E., M.I.M.E. 14 *Leaside Avenue, Muswell Hill*, N.
June 22, 1894.	‖Larmor, Prof. Sir Joseph, M.A., D.Sc., F.R.S. *St. John's College, Cambridge.*
May 27, 1876.	‖Law, Channell. *The Museum, Torquay.*
Jan. 25, 1907.	Lean, Clement, B.Sc., A.M.I.Mech.E. *Thanet House*, 231 *Strand*, W.C.
Feb. 8, 1907.	§†Lees, Prof. C. H., D.Sc., F.R.S. *Greenacres, Woodside Road, Woodford Green, Essex.*
Nov. 22, 1895.	§‖Lehfeldt, R. A., D.Sc. *Transvaal Technical Institute, Johannesburg, S. Africa.*
Feb. 28, 1896.	Lewis, Prof. D. Morgan, M.A. *Iscoed, Caradog Road, Aberystwyth.*
Feb. 14, 1880.	‖Lewis, T. C., M.A. *West Home, West Road, Cambridge.*
Jan. 17, 1882.	‖Lewis, Prof. W. J., M.A., F.R.S. *Mineralogical Museum, Cambridge.*
June 24, 1904.	Liebert, W. *C/o Messrs. Huttenbach Bros. & Co., Penang, India.*
May 26, 1916.	Lingen, J. S. van der. *Seapoint, Cape Town.*
Feb. 9, 1917.	Lister, John. 50 *Bradbourne Street, Parson's Green*, S.W.
June 14, 1895.	Littlewood, T. H., M.A. *Bancroft's School, Woodford Wells, Essex.*
Feb. 22, 1879.	‖Liveing, Prof. G. D., M.A., F.R.S. *Newnham, Cambridge.*
Mar. 25, 1876.	‖Liversidge, Prof. A., M.A., F.R.S. *Fieldhead, George Road, Coombe Warren, Kingston.*
May 8, 1914.	Lob, R., B.A. 16 *Unwin Mansions, Queen's Club Gardens*, W
Feb. 3, 1877.	Lockyer, Sir J. Norman, K.C.B., F.R.S. *Hill Observatory, Salcombe Regis, Devon.*
Feb. 13, 1875.	‖LODGE, Principal Sir O. J., D.Sc., F.R.S. (PAST PRESIDENT) *Mariemont, Edgbaston*, and *The University, Birmingham.*
Dec. 11, 1903.	Lough, Victor. *Glendale, High Road, Loughton, Essex.*
Feb. 23, 1884.	‖Love, E. F. J., M.A. *The University, Melbourne, Australia.*
Nov. 27, 1896	Lowater, Miss Fanny, B.Sc., Ph.D. *Wellesley College, Wellesley, Mass., U.S.A.*
Mar. 27, 1903.	Lownds, Louis, B.Sc., Ph.D. *South-West London Polytechnic, Manresa Road, Chelsea*, S.W.
Mar. 6, 1891.	Lucas, William, M.A. 40 *Rosebery Gardens, Crouch End*, N.
	§†*‖Lupton, Sydney, M.A. 102 *Park Street, Grosvenor Sq.*, W.
Jan. 23, 1903.	Lyle, Prof. T. R., Sc.D., F.R.S. *The University of Melbourne, Victoria, Australia.*
Jan. 25, 1879.	Macdonell, Prof. A., B.A. *The Abbey House, Shaftesbury, Dorset.*

Date of Election.	
May 20, 1915.	McDowall, Rev. Stewart A. *The College, Winchester.*
June 14, 1901.	McEwen, J. M. 30 *Borneo Street, Putney.*
Feb. 14, 1913.	McKinnon, D., B.Sc. 78 *Henniker Gardens, East Ham*, E.
May 11, 1878.	‖Magnus, Sir Philip, B.Sc., M.P. 16 *Gloucester Terrace, Hyde Park*, W.
Oct. 30, 1896.	‖Mair, D. B., M.A. *Civil Service Commission, Burlington Gardens*, W.
Feb. 22, 1901.	Makower, Walter, D.Sc. *Maylands, Brook Road, Fallowfield, Manchester.*
Nov. 18, 1876.	‖Malcolm, Col. E. D., R.E., C.B. *Poltalloch, Loch Gilphead, Argyll.*
Jan. 23, 1914.	Malcolm, H. W., D.Sc. *Gidea Lodge, Reed Pond Walk, Gidea Park, Romford, Essex.*
Mar. 7, 1890.	‖Mance, Sir Henry, LL.D. 25 *Linton Road, Oxford.*
June 28, 1912.	Manquais, J. C. R. Le, A.R.C.S. 101 *Clifton Street, Swindon, Wilts.*
Mar. 11, 1910.	Marsden, E., D.Sc. *The Physical Laboratory, Victoria University College, Wellington, N.Z.*
Mar. 8, 1907.	Martyn, G. H., B.Sc. 38 *Hogarth Hill, Garden Suburb, Hampstead*, N.W.
Feb. 24, 1911.	Mason, C. C., M.A. *Beverley, Cavendish Avenue, Cambridge.*
Mar. 23, 1889.	‖Massey, W. H., M.Inst.C.E. *Twyford, R.S.O., Berkshire.*
May 14, 1887.	†§‖Mather, Prof. T., F.R.S. 37 *Wyatt Park Road, Streatham Hill*, S.W.
June 22, 1900.	Mayall, R. H. D., M.A. *Sidney Sussex College, Cambridge.*
Mar. 6, 1891.	‖Miers, Principal Sir H. A., D.Sc., F.R.S. *Birch Heys, Cromwell Range, Fallowfield, Manchester.*
Feb. 13, 1886.	‖Mills, A. E., M.A., M.Inst.E.E. *Halcyon, North Finchley*, N.
	§**‖Mills, Prof. E. J., F.R.S. 64 *Twyford Avenue, West Acton*, W.
Dec. 4, 1891.	‖Mond, Robert Ludwig, B.A., F.G.S. *Coombe Bank, near Sevenoaks, Kent.*
Jan. 26, 1912.	Moore, Harry. 21 *Waldegrave Road, Hanger Lane, Ealing*, W.
Apr. 13, 1889.	Mordey, W. M., M.Inst.C.E., M.I.E.E. 82 *Victoria Street*, S.W.
May 12, 1877.	‖Morley, H. F., M.A., B.Sc. 5 *Lyndhurst Rd., Hampstead*, N.W.
Feb. 24, 1911.	Moss, E. W. 12 *Windermere Avenue, Church End*, N.
Mar. 22, 1895.	‖Morris, Dr. D. K. *Messrs. Morris & Lister, Ltd., Carlton Works, Coventry.*
May 11, 1906.	Morrow, John, M.Sc., D.Eng. *Armstrong College, Newcastle-on-Tyne.*
Mar. 22, 1895.	‖Morton, Prof. W. B., M.A. *Queen's College*, and *Glencarse, Notting Hill, Belfast.*
Mar. 8, 1895.	‖Mostyn, S. G., M.A. *Health Office, Houndgate, Darlington.*
June 28, 1879.	Moulton, Rt. Hon. Lord, M.A., F.R.S. 57 *Onslow Square*, S.W.
Nov. 28, 1913.	Mourad, Lieut. S.
Mar. 23, 1889.	‖Mudford, F. J. *Arrow Electrical Works, Jackson Road, Holloway*, N.
	**‖Muirhead, Alexander, D.Sc., F.R.S. *The Lodge, Shortlands, Kent.*
Mar. 12, 1909.	Narbeth, B. M. *Technical Institute, Durban, Natal, S. Africa.*

Date of Election.	
Dec. 11, 1886.	Natanson, Ladislas. *Professor, Cracow University*; 3 *Studencka str., Cracow* (*Austria*).
Nov. 26, 1909.	Nettleton, H. R., B.Sc. *Birkbeck College, Bream's Buildings, Chancery Lane*, E.C.
Apr. 8, 1910.	Neville, B. M. *Silverleigh*, 42 *Constantine Road, Hampstead*, N.W.
Feb. 25, 1888.	‖Newall, Prof. H. F., M.A., F.R.S. *Madingly Rise, Cambridge*.
Nov. 12, 1909.	NICHOLSON, Prof. J. W., M.A., D.Sc. (MEMBER OF COUNCIL). *King's College, Strand*, W.C.
Nov. 11, 1910.	Nicol, James. 15 *Lansdowne Road, Muswell Hill*, N.
Nov. 12, 1881.	Niven, Sir W. D., M.A., F.R.S. *Eastburn, St. John's Road, Sidcup*.
Nov. 25, 1898.	Noble, Wilson. *Park Place, Henley-on-Thames*.
May 8, 1914.	Norman, Sir Henry, Bart., M.P. *Honeyhanger, Haslemere, Surrey*
Apr. 26, 1895.	‖Notcutt, S. A., LL.M., B.A., B.Sc. *Constitution Hill, Ipswich*.
Nov. 14, 1913.	Nottage, W. H., B.Sc. 29 *London Road, Chelmsford*.
Oct. 23, 1914.	Oba, Prof. Shigeni. *Kiushu University, Japan*.
Apr. 10, 1875.	‖Odling, Prof. W., M.A., F.R.S. 15 *Norham Gardens, Oxford*.
Dec. 9, 1904.	Owen, D., D.Sc. 39 *Dalebury Road, Upper Tooting*, S.W.
Nov. 22, 1912.	Owen, E. A. *Highfield, Stanley Road, Teddington*.
Feb. 11, 1916.	Owen, Prof. G., D.Sc. *Auckland University College, N.Z.*
May 31, 1912.	Owen, Morris. *Victoria College, Nagpur, C.P., India*.
July 8, 1910.	§O'Meara, Major W. A. J., C.M.G. 22 *Talgarth Road, West Kensington*, S.W.
May 27, 1904.	Parker, F. H., B.Sc. 72 *Greenvale Road, Eltham, Kent*.
Feb. 8, 1907.	PATERSON, C. C., A.M.Inst.C.E. (MEMBER OF COUNCIL). *National Physical Laboratory, Teddington*.
	**‖PERRY, Prof. J., D.Sc., F.R.S. (PAST PRESIDENT). 25 *Stanley Crescent, Kensington*, W.
May 25, 1878.	‖Phelips, W. R. *Montacute House, Ilminster*.
Feb. 24, 1899.	PHILLIPS, Capt. C. E. S., F.R.S.E. (MEMBER OF COUNCIL). *Royal Herbert Hospital, Shooter's Hill, Kent*.
Apr. 22, 1904.	Phillips, W. C. S., B.Sc. *Central School of Science and Technology, Victoria Road, Stoke-upon-Trent*.
Mar. 13, 1886.	‖Pickering, Prof. S. U., M.A., F.R.S. *Harpenden, Herts*; and 60 *Palace Court*, W.
Jan. 16, 1891.	‖Pidgeon, Winter Randall, M.A. *Lynsted Lodge, St. Edmund's Terrace, Regent's Park*, N.W.
Apr. 24, 1903.	Polley, L. J. N. *Rutlich School, Merton*, S.W.
May 12, 1899.	‖Pollock, Prof. J. A., B.Sc., F.R.S. *University of Sydney, N.S.W.*
Jan. 16, 1891.	§Porter, Alfred William, D.Sc., F.R.S. *University College, Gower Street*, W.C.
Nov. 11, 1892.	§‖Porter, T. C., M.A., D.Sc. *Upton Park, Slough*.
Mar. 16, 1878.	‖Prance, Miss Edith. *The Ferns, Frognal Lane, Hampstead*, N.W.

Date of Election.	
May 8, 1886.	§‖Price, W. A., M.A. 135 *Sandyford Road, Newcastle-on-Tyne.*
May 30, 1913.	Prior, A., B.S. 63 *Church Street, Stoke Newington*, N.
Dec. 9, 1892.	Rawson, Col. Herbert E., C.B., R.E., J.P. *Home Close, Heronsgate, Herts.*
Dec. 15, 1877.	§†‖Rayleigh, The Right Hon. Lord, O.M., M.A., D.C.L., F.R.S. *Terling Place, Witham, Essex.*
Feb. 8, 1907.	‖Rayner, E. H. *Tiviot Dale, Gloucester Road, Teddington.*
Mar. 10, 1911.	Read, R. S., M.A., B.Sc., A.R.C.Sc. 38 *Radford Road, Hither Green, Lewisham*, S.E.
	**‖REINOLD, Prof. A. W., C.B., M.A., F.R.S. (Past President). 11 *Spencer Road, Southsea.*
Feb. 13, 1903.	Rheinberg, J. 23 *The Avenue, Brondesbury Park*, N.W.
May 8, 1903.	RICHARDSON, Prof. O. W., M.A., D.Sc., F.R.S. (Member of Council). 4 *Cannon Place, Hampstead*, N.W.
June 12, 1886.	‖Rideal, S., D.Sc. 28 *Victoria Street, Westminster*, S.W.
Nov. 9, 1894.	Rintoul, D., M.A. 26 *College Road, Clifton, Bristol.*
Mar. 11, 1910.	Robinson, James, M.Sc., Ph.D. *C/o W. Buckley, Tynemouth Road, N. Shields.*
Jan. 24, 1885.	‖Rose-Innes, J., M.A., B.Sc. *King's College, Cambridge.*
Apr. 26, 1907.	RUSS, Sidney, D.Sc. (Member of Council). *Cancer Research Laboratories, Middlesex Hospital*, W.
Jan. 26, 1906.	§†Russell, Alexander, M.A., D.Sc. 1 *Richmond Bridge Mansions, East Twickenham, Middlesex.*
Jan. 19, 1878.	‖Russell, J. W., M.A. 28 *Staverton Road, Oxford.*
Mar. 22, 1907.	§Rutherford, Prof. Sir Ernest, D.Sc., F.R.S. *The University, Manchester.*
Jan. 15, 1876.	‖Salomons, Sir D. L., Bart. *Broom Hill, Tunbridge Wells.*
Jan. 30, 1875.	‖Salter, M. J. *Greyrick House, Mickleton, Gloucestershire.*
May 13, 1898.	Sampson, J. *University, Liverpool.*
June 27, 1885.	‖Sanderson, F. W., B.A. *The School House, Oundle.*
Feb. 9, 1917.	Satterley, John, D.Sc., M.A. *University of Toronto, Physics Buildings, Toronto.*
Dec. 9, 1882.	‖Saunders, Rev. S. T. H., M.A. 39 *Buckland Crescent*, N.W.
Nov. 26, 1909.	Schofield, H. *The Technical Institute, Loughborough.*
Feb. 28, 1902.	Schofield, J. *Rose Cottage, Netherton, Huddersfield.*
	†**‖SCHUSTER, Prof. A., Ph.D., D.Sc., LL.D. F.R.S. (Past President). *Yeldall, Twyford, Berks.*
Apr. 26, 1907.	Scoble, Walter A., A.R.C.S., B.Sc. *Finsbury Technical College, Leonard Street*, E.C.
Feb. 21, 1883.	Scott, Prof. A.W. *St. David's College, Lampeter, South Wales.*
Feb. 24, 1911.	Sears, J. E., Junr. *National Physical Laboratory, Teddington, Middlesex.*
Dec. 19, 1894.	§‖Searle, G. F. C., M.A., D.Sc., F.R.S. *Wyncote, Hills Road, Cambridge.*
Mar. 17, 1877.	‖Seaton, W. S. *Montague House, Richmond, Surrey.*
May 9, 1885.	‖Selby, Prof. A. L., M.A. 29 *Palace Road, Llandaff, Cardiff.*
June 14, 1901.	Selby, F. J., M.A. *Ravensbourne, Gloucester Road, Teddington, Middlesex.*

Date of Election.	
Jan. 23, 1914.	Shaw, Herman. 5 *Dartmouth Park Road, Highgate*, N.W.
Jan. 8, 1895.	Shaw, P. E., B.A., D.Sc. *University College, Nottingham.*
Mar. 26, 1906.	Shaw, Walter, B.Sc. 1 *Bute Road, Barking Side, Ilford.*
Feb. 8, 1879.	‖SHAW, Sir Napier, M.A., ScD., F.R.S. (Vice President). 10 *Moreton Gardens, South Kensington*, S.W.
Feb. 13, 1914.	Shaxby, J. H. *Wynnston, Llanishen, Glamorgan.*
Mar. 7, 1890.	‖Shorter, Lewis, R., B.Sc. *Bowland, Rickmansworth, Herts.*
May 22, 1914.	Shrimpton, A. G. 75 *Keslake Road, Kensal Rise, Kilburn*, N.W.
Oct. 25, 1895.	‖Shurlock, F. W., B.A., B.Sc. 3 *Lime Avenue, Derby.*
Apr. 13, 1889.	‖Siemens, A. *Palace Place Mansions, Kensington Court*, W.
Mar. 9, 1917.	Simeon, Fredk., B.Sc. *Care of* Adam Kilger, Ltd., 75A *Camden Road*, N.W.
Jan. 22, 1881.	‖Simpson, G. P., B.Sc. *Board of Trade, Exhibitions Branch, Queen Anne's Chambers, Westminster*, S.W.
Dec. 10, 1881.	‖Sircom, Prof. the Rev. Sebastian, M.A. *Care of* Rev. Marchant, *St. Mary's, Stonyhurst, Blackburn.*
Dec. 14, 1894.	§†‖Skinner, S., M.A. *South-west London Polytechnic, Manresa Road, Chelsea*, S.W.
Dec. 13, 1895.	‖Slater, R. C., M.A. *Charterhouse, Godalming, Surrey.*
Nov. 23, 1878.	‖Sleeman, Rev. P. R., F.R.A.S. 65 *Pembroke Road, Clifton, Bristol.*
Mar. 22, 1901.	Smart, E., B.A., B.Sc. *Tillyloss, Tullylumb Terrace, Perth.*
June 10, 1904.	Smith, E., B.Sc. "*Glenholme*," *Colchester Road, Leyton*, E.
Feb. 7, 1890.	‖Smith, Edward Wythe. 19 *Newburgh Road, Acton.*
Feb. 8, 1907.	§†Smith, F. E., A.R.C.S. *Redcot, St. James' Avenue, Hampton Hill, Middlesex.*
Jan. 24, 1908.	Smith, R. Stuart, B.A. 58 *Burton Road, Melton Mowbray.*
Nov. 26, 1897.	SMITH, S. W. J., M.A., D.Sc., F.R.S. (Vice-President and Librarian). *Imperial College of Science & Technology, S.Kensington ;* 12 *Fordhook Avenue, Ealing*, W.
Mar. 12, 1909.	SMITH, Thomas, B.A. (Member of Council). *National Physical Laboratory, Teddington.*
Apr. 10, 1880.	‖Smith, W. O. *Century Works, Lewisham*, S.E.
June 26, 1896.	‖Sodeau, W. H., D.Sc. *Torpedofabrik, Fiume, Hungary.*
June 8, 1878.	‖Solly, R. H., M.A., F.G.S. *Care of* A. Hutchinson, *Pembroke College, Cambridge.*
Feb. 13, 1914.	South, H. J. 30 *Colombo Road, Ilford, Essex.*
Feb. 24, 1899.	Sowter, R. J., B.Sc. "*Carleton*," *The Avenue South, Surbiton.*
May 8, 1914.	Squier, Lt.-Col. George O. *War Department, Washington, D.C., U.S.A.*
Mar. 8, 1895.	‖Stansfield, Prof. H., D.Sc. *Hartley University College, Southampton.*
Apr. 23, 1898.	Starling, S. G., B.Sc. *Municipal Technical Institute, Romford Road, West Ham.*
June 11, 1881.	‖Stead, J. E., F.R.S. *Redcar.*
June 20, 1914.	Stephenson, J., B.Sc., A.R.C.S. *Bankfield, Linthwaite, near Huddersfield.*
June 28, 1895.	‖Stewart, J. J., M.A., B.Sc. 2 *Stow Park Crescent, Newport, Monmouthshire.*
	*Stirling, Hon. Sir James, M.A., F.R.S. *Finchcocks, Goudhurst, Kent.*

Date of Election.	
Dec. 15, 1877.	‖Stocker, Prof. W. N., M.A. *Brasenose College, Oxford.*
Nov. 27, 1914.	Stoney, Miss E. A. 20 *Reynolds Close, Hampstead Way, London,* N.W.
Nov. 13, 1914.	Stratton, F. J. M. *Gonville and Caius College, Cambridge.*
June 9, 1893.	‖Stroud, Prof. Henry, M.A., D.Sc. *Armstrong College, Newcastle-on-Tyne.*
Feb. 26, 1887.	§‖Stroud, Prof. W., D.Sc. *Caxton Street, Anniesland, Glasgow.*
Mar. 12, 1909.	§‖Strutt, Prof. the Hon. R. J., F.R.S. 69 *Cadogan Square,* S.W.
Dec. 10, 1887.	‖SUMPNER, W. E., D.Sc. (VICE-PRESIDENT). *Municipal Technical School, Suffolk Street, Birmingham.*
Nov. 23, 1906.	Swann, W. F. G., D.Sc., A.R.C.S. *Carnegie Inst.* (*Washington*), 36*th Street and Broad Branch Road, Washington, U.S.A.*
May 22, 1886.	§†‖Swinburne, J., F.R.S. 82 *Victoria Street,* S.W.
Feb. 8, 1895.	§Swinton, A. A. Campbell, M.Inst.C.E., F.R.S. 66 *Victoria Street,* S.W.
Nov. 16, 1916.	Thirunal, Rajah Rohani, F.R.Met.Soc. *Kizhakke Kottaram Palace, Trivandrum, Travancore, S. India.*
May 28. 1897.	Thomas-Davies, G. *Gilwern, Pontllanfraith, Mon.*
Jan. 26, 1917.	Thompson, A. J., B.Sc. "*Bronia,*" *Link Lane, Wallington, Surrey.*
Mar. 22, 1895.	‖THOMSON, Prof. Sir J. J., O.M., M.A., D.Sc., LL.D., Ph.D., F.R.S. (PAST PRESIDENT). *Holmleigh, West Road, Cambridge.*
Mar. 22, 1907.	Thomson, W., M.A. 47 *Nassau Road, Barnes,* S.W.
Nov. 28, 1885.	‖Thorpe, Prof. Sir Edward, Ph.D., F.R.S. *Whinfield, Salcombe, South Devon.*
Mar. 9, 1889.	§‖Threlfall, R., M.A., F.R.S. *Oakhurst, Church Road, Edgbaston, Birmingham.*
Jan. 26, 1900.	Tomkins, J. A. *Wesley Villa, Town Street, Bramley, Leeds.*
	§*‖Tomlinson, Herbert, B.A., F.R.S. *Gilling, Upper Station Road, Bexhill.*
June 8, 1900.	Tomlinson, H. J. 56 *Harold Road, Upper Norwood.*
June 26, 1903.	Townsend, Prof. J. S., M.A., F.R.S. *The Electrical Laboratory, Oxford.*
Dec. 8, 1888.	‖Trotter, Alexander Pelham, B.A. 8 *Richmond Terrace, Whitehall,* S.W.
Mar. 8, 1895.	§Trouton, Prof. F. T., M.A., Sc.D., F.R.S. *The Rydings,* 35 *Redington Road, Hampstead,* N.W.
Feb. 13, 1903.	Tucker, W. S., B.Sc. 115 *Northbrook Road, Cranbrook Road, Ilford.*
Feb. 22, 1901.	Tutton, A. E. H., M.A., D.Sc. (Oxon), F.R.S. *Duart, Yelverton, S. Devon.*
Nov. 25, 1904.	§Twyman, F., A.I.E.E. *Messrs. A. Hilger,* 75A *Camden Road,* N.W.
Mar. 8, 1912.	Tyndall, A. Mannering, D.Sc., M.Sc. 9 *Henleare Gardens, Bristol.*
	§**‖Unwin, Prof. W. C., B.Sc., F.R.S. *Palace-Gate Mansions,* 29 *Palace Gate, Kensington,* W.

Date of Election.	
Nov. 22, 1895.	‖Vincent, J. H., D.Sc., M.A. 20 *Mostyn Avenue, Wembley, Middlesex.*
Apr. 14, 1905.	Vinter, P. J., M.A. *Archbishop Holgate's School, York.*
Feb. 13, 1902.	Wagstaff, C. J. L. 62 *Westbere Road, Cricklewood*, N.W.
Dec. 1, 1883.	§†‖Walker, J., M.A. 30 *Norham Gardens, Oxford.*
Jan. 23, 1914.	Wall, W. G. P., M.Sc. *Muir Central College, Allahabad.*
Mar. 7, 1890.	‖Waller, Augustus D., M.D., LL.D., F.R.S. 32 *Grove End Road*, N.W.
Nov. 24, 1888.	‖Walmsley, R. M., D.Sc. *Northampton Institute*, E.C.
Dec. 11, 1891.	§†‖Watson, William, D.Sc., F.R.S. 7 *Upper Cheyne Row*, S.W.
Jan. 23, 1914.	Welch, A. P. *Greenlea, Ealing Road, Wembley.*
Dec. 11, 1886.	‖Wertheimer, Prof. Julius, B.A., D.Sc. *Merchant Venturers' Technical College, Bristol.*
Mar. 8, 1912.	West, Gilbert D., B.Sc. 105 *The Drive, Fulham Park Gardens*, S.W.
Dec. 14, 1894.	§‖Whetham, W. C. D., M.A., F.R.S. *Trinity College, Cambridge.*
Feb. 26, 1915.	WHIPPLE, F. J. W. (MEMBER OF COUNCIL). 6 *Addison Road, Bedford Park*, W.
Dec. 9, 1889.	§†Whipple, R. S. *Wexcombe, Luard Road, Cambridge.*
May 10, 1895.	§‖White, Prof. S. A. F., M.A. *King's College*, W.C.
June 8, 1906.	White, W. H. 24 *Silent Street, Ipswich.*
Feb. 24, 1899.	Whitehead, C. S., M.A. 3 *St. Catherine's Terrace, Hove, Sussex.*
Dec. 15, 1894.	§‖Wilberforce, Prof. L. R., M.A. *The University, Liverpool.*
Mar. 10, 1905.	Wilkinson, Ambrose, B.Sc. *University College, Nottingham.*
May 26, 1916.	Williams, H. B., A.B., M.D. 437 *W. 59th Street, New York City.*
Jan. 27, 1907.	WILLOWS, R. S., M.A., D.Sc. (SECRETARY). *Cass Technica Institute, Jewry Street, Aldgate*, E.C.; 15 *Rodenhurst Road, Clapham Park*, S.W. 4.
June 12, 1903.	§‖Wilson, Prof. H. A., M.A., D.Sc., F.R.S. *The Rice Institute, Houston, Texas, U.S.A.*
Jan. 26, 1912.	Wilson, W., Ph.D. 33 *Cloudesdale Road, Upper Tooting*, S.W.
Dec. 15, 1877.	‖Wood, Sir H. T., B.A. *Society of Arts, Adelphi*, W.C.
	§*‖Wormell, Dr. R. *Roydon, nr. Ware.*
May 14, 1909.	Wright, Arthur, M.I.E.E. 3 *Addison Road*, W.
Dec. 13, 1880.	‖Wrightson, Sir T., C.E. *Neasham Hall, Darlington.*
Nov. 10, 1916.	Young, F. B., B.A., M.Sc. 2 *Eastfield Road, Cotham, Bristol.*
Feb. 27, 1886.	§‖Young, Prof. Sydney, D.Sc., F.R.S. *University Chemical Laboratory, Trinity College, Dublin; and* 12 *Raglan Road, Dublin.*

Students.

May 14, 1915.	Adams, B. A. 118 *High Street, Beckenham.*
Jan. 23, 1914.	Bessent, Percy. 47 *Barclay Road, Fulham*, S.W.
Feb. 12, 1915.	Edwards, D. L. 18 *The Avenue, Hitchin.*
Feb. 11, 1916.	Hayes, Cecil, B.Sc. *Melhurst, Mona Street, Beeston, Notts.*
Feb. 12, 1915.	Hopkins, David G. 43 *Rhondda Street, Swansea.*
Dec. 18, 1914.	Schroeder, F. H. 138 *Great Portland Street, London*, W.
Feb. 11, 1916.	Warren, Arthur, B.Sc. 23 *The Wardwick, Derby.*

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In order to overcome this difficulty, brass cylinders, each 8 cm. long, were accurately turned, so that the inside and outside surfaces were concentric and the inner diameter equal to that of the copper wire. A small longitudinal opening was made in each cylinder, through which the wire was passed, and attached with a little solder. Any slight rotation of the lead spheres will now introduce no error, seeing that the axis of rotation is the geometric axis of the outside surface.

The method of taking the readings was as follows: The micrometer readings for the two wires were taken for the A position and then for the B position (Part I. in table). The temperature of the lead spheres was then raised to 220°C., and the four micrometer readings again taken (Part II. in table). This was again repeated when the spheres were cold (Part III. in table). In order to be certain that the torsion system was moving freely the scale reading of the torsion beam was taken roughly for each of the above positions.

To assist in keeping the position of the steel rods fixed during the experiment the four pieces of T-iron were wrapped with felt, cotton and cardboard, and the four wooden pillars were bound together by other pieces of T-iron, which were similarly lagged to prevent appreciable rise of temperature during the experiment.

4. *Readings.*

The readings are shown in the table on next page. In the first column is the position of the lead balls, while in the second is the tube reading. The latter is obtained by the mirror which is attached to the vacuum tube, which records any rotation of the apparatus as a whole. In the third column are the readings of the position of the torsion system. This was never at rest, but from two consecutive readings of the extreme positions the exact position of rest can be calculated if the decrement of the system be known. In the next column we have the temperature of the lead balls.

A wooden pillar is referred to as the north, south, east or west pillar, according to its position with respect to the cardinal points of the compass. Thus in the A position of the lead balls we have readings of the micrometer from the east and west pillars, while for the B position they are from the north and south pillars. Since a movement of the wire towards the vacuum tube will give *a smaller* micrometer reading, it will be seen that the *sum* of the east and west (or north and south)

Table of Results.

	Pos.	T.R. mm.	R. mm.	θ_L.	Radial readings of M.M. in mm.					
					N.	S.	E.	W.	N+S.	E+W.
I.	B	... 324·9	237·6 382·7	... 19°C.	6·516 6·525	... 15·841	... ...	... ...	... ...	... ...
	A	324·3	220·4 64·9	... 19°C.	... ...	... ...	12·350 12·360	10·310 10·303	... ...	... ...
	B	...	...	19°C.	6·527	15·845	...	...	...	...
					6·523	15·843	12·355	10·307	22·366	22·662
II.	A	324·0	53·9 173·9	216°C. 202°C.	... ...	... ...	... 12·385	... 10·284	... ...	... ...
	B	...	...	206°C. 190°C.	6·540 6·542	15·862 15·855	... ...	... ...	... ...	... ...
	A	...	...	199°C. 182°C.	... ...	... ...	... 12·415	... 10·267	... ...	... ...
	B	324·8	404·1 268·9	190°C. 176·C.	... 6·540	... 15·858	... ...	... ...	... ...	... ...
					6·541	15·858	12·400	10·275	22·399	22·675
III.	B	324·9	433·1 249·9	... 23°C.	... 6·535	... 15·830	... ...	... ...	... 22·365	... ...
	A	324·4	9·7 204·6	... 23°C.	... ...	... ...	... 12·340	... 10·327	... ...	... 22·667
Mean for cold									22·366	22·665
,, hot									22·399	22·675
Difference									+0·033	+0·010
Mean difference = +0·021 mm.										

readings, together with some unknown *constant* length will give us the distance between the wires for the A (or B) position of the lead balls. The difference in the numbers in the ninth and tenth columns are due to the chance that the vertical pins for supporting the micrometer are not equidistant from the central axis.

From these results it will be seen that for the A position we have an OUTWARD movement of 0·01 mm. and for the B position an OUTWARD movement of 0·03 mm., giving a mean outward movement of 0·02 mm.

5. *Result.*

As pointed out earlier the temperature effect observed in the original research could be produced if an INWARD movement of 0·15 mm. occurred in *each* lead sphere.

The present test shows that the total movement of *both* lead spheres is 0·02 mm., so that *each* lead sphere would only move

0·01 mm. This is one-fifteenth of the effect looked for, and it is *outward—i.e.*, in the contrary sense. Thus, the gravitation-temperature effect will now have to be increased slightly.

Before calculating the correction to be applied to the temperature coefficient for this small observed displacement, we notice that it would require a rise in temperature of only 4°C. in the turn-table beam to produce the outward movement of 0·02 mm. It is not unlikely that the beam would rise in temperature by this amount, and we feel inclined to attribute the effect observed in this test to this cause rather than to any Bernouilli effect or radiation pressure.

In the original research the probable result is

$$a=+(1{\cdot}2\pm0{\cdot}05)\times10^{-5}.$$

As a result of the present test this must now be amended to—

$$a=+(1{\cdot}3\pm0{\cdot}05)\times10^{-5}.$$

In conclusion, we might say that the present method, rather than an optical one, was chosen chiefly because of the difficulty of working any optical device with the gravitation apparatus as already installed.

In order to avoid small errors of displacement it seems advisable always in future to apply a test for the horizontal distance between the suspending wires in making gravitation-temperature experiments, and we are inclined in future to give optical methods a trial as the above method is rather tedious. We find that mechanical contact of the micrometer screw on the suspending wires tends to set these in vibration; whereas if a microscope or an optical lever device were used there would, of course, be no actual contact or, in the latter case, only a very slight one.

ABSTRACT.

In the Philosophical "Transactions" of the Royal Society, Vol. CCXVI., pp. 349 to 392, there is a Paper by one of the authors dealing with the possible existence of a temperature coefficient of the constant of gravitation. It was suggested in the discussion that the effect might be due to an inward displacement of the large lead spheres, at the higher temperatures, due to convection currents.

In the present Paper experiments are described in which this point is tested by micrometric measurements of the positions of the supporting wires. It is shown that, at the higher temperatures, there is a small *outward* displacement of the spheres, probably due to the expansion of the crosshead from which they are suspended. A slightly ***higher*** value has, therefore, to be given to the temperature coefficient of gravitation.

DISCUSSION.

Mr. F. E. **Smith** suggested that since it appeared to be necessary to measure the deflection of the large spheres in order to arrive at the correct result, this could be more advantageously done with an arrangement of microscopes than with the micrometers employed.

Prof. C. V. **Boys** thought the subject was one of vital importance. He was not yet convinced that the results obtained by Dr. Shaw were necessarily true; but the apparatus was in many ways so beautifully designed that it was difficult to see what could give rise to the discrepancy. It seemed to him that Dr. Shaw's method of measuring the constant at two different temperatures was not the best way of measuring the *difference* between these values, which was thus subject to the inaccuracy of the complete determination. It seemed a better plan to have two sets of balls, one in the A position and one in the B position, and to measure any change in the deflection produced when one pair were hot and the other cold, and *vice versa.* This would be a null method, and would measure only the temperature coefficient if such existed. He had mentioned this to Dr. Shaw, who had brought forward some difficulty in connection with it which he could not at the moment remember.

Mr. C. C. **Paterson** asked if it was not possible for convection currents to be set up inside the vacuum tube and to affect the small spheres.

Dr. **Shaw** replied as follows: Mr. Smith would prefer a set of microscopes to a set of electric micrometers for the measurements. We state at the end of the Paper that we should have tried, and very likely adopted, this method if we had been able; but the scaffolding and permanent tables surrounding the gravitation apparatus prevented that free access which would have been required for optical measurement. We should on another occasion make attempts to use a microscope, though, of course, there are special difficulties (*e.g.*, vibration) involved in its use on a long wire. The alternative method suggested by the President, ideal in some experiments, would seem to suffer in comparison with the method used in two respects. (1) The observed temperature effect would be reduced to one-half, since the hot sphere would, in effect, act on one side only, instead of acting, as at present, on both sides. The effect, already small, would thus be measurable with only half the accuracy. (2) A more serious technical trouble would arise from the complete inability to check the zero of the torsion beam. This zero may drift in the method so far used, without directly invalidating the result; but in the proposed method any drift would enter the result as a direct error. In the method used we can observe zero continually, and the constancy of the zero really affords some indication of the accurate working of the apparatus. The possibility of movements of gas in the vacuum must be kept in mind, as Mr. Paterson urges. The arguments against the occurrence of these movements in the experiments are given in the original Paper. The hot lead spheres are separated from the vacuum chamber containing the torsion system by a screen of flowing water at constant temperature, and by a composite lagging of cotton wool, flannel, &c. The only agency which could cause gas movements would be the *one* factor which changes during the experiment—viz., the heat introduced into the lead. This heat on leaving the lead would have to pass in succession through cotton wool, flowing water, cotton wool, glass tubing, silver tubing. On reaching the inner wall of the silver tubing it would be liberated to the gas in the vacuum chamber. Now, the cycle from the A position to the B position and back to the A position takes only 12 minutes. Is it conceivable that the supposed flow of heat from the lead to the gas can be so abundant and quick through the composite screen that the gas movement can reverse its direction in six minutes? But, apart from conjecture, actual tests were made of the effect when the lead spheres were kept a very short time at the A and B position. No time effect could be observed. Again, the lagging was varied greatly in thickness without observable effect. The test described in this Paper is the only one I have been able to make since 1915, on account of the practical impossibility of obtaining necessary material (copper or tungsten) at present.

XI. *To Measure the Pressure in a High Vacuum by Observation of Logarithmic Decrement.* *By* P. E. Shaw, *B.A.*, *D.Sc.*

Received January 9, 1917.

In experiments on the Newtonian constant ("Phil. Trans.," May, 1916) I used a torsion balance, in a vacuum which varied in different cases from 15 mm. to 0·01 μ pressure. Before the vacuum vessel is sealed off the pressure is determined by a McLeod gauge, but after sealing off there does not seem to be any ready, accurate method in general use for this determination. But by observing the damping of the vibration of the torsion system in *high* vacuum, we have a means at our disposal. We merely use the ordinary readings of amplitude, and no special readings or special apparatus are required for the determination of pressure.

For a vacuum pressure of the order of 1 μ (=0·001 mm.) of mercury the mean free path of oxygen, the gas in use, is at 0° C. of the order 10 cm. This is large compared with the diameter of the vacuum vessel, and the size of the suspended bodies, so that the density of the mass coming up to the suspended body is unaffected by the motions. For almost every particle reaching the body comes from the bounding surface of the vacuum vessel, and its velocity will depend on the temperature of the bounding surface, so that the distribution of paths will be always equally irregular.

Divide the mass of gas particles in front of the suspended body into six streams. The stream moving towards the body, supposed a flat plane, has velocity V, say $\longrightarrow$; the stream moving away from it has velocity $V+2v \longleftarrow$, where v is velocity of the plane, and the mass moving to the plane per unit area per second$=\rho(V+v)/6$, ρ being the gas density.

The momentum imparted per second, *i.e.*, the pressure, is :—

$$\rho(V+v)2(V+v)/6.$$

This is practically

$$\rho(V^2+2Vv)/3.$$

On the back face the pressure is :—

$$\rho(V^2-2Vv)/3.$$

The excess on the front face is :—

$$4\rho Vv/3.$$

If s=area of the plane, and a=its arm, $v=a\dot{\theta}$; and we have as equation of motion :—

$$I\ddot{\theta}+\tfrac{4}{3}\rho Vsa\dot{\theta}+\mu\theta=0.$$

Putting this as :—

$$\ddot{\theta}+K\dot{\theta}+n^2\theta=0,$$

where $K=4\rho Vsa/3I$.

Solving the above differential equation we obtain :—

$$\text{log dec., } \lambda=KT/4,$$

where T=periodic time.

In September, 1910, when discussing my gravitation research with the late Prof. J. H. Poynting, I said that I thought of measuring the vacuum by observations of damping. Next day Prof. Poynting, with his well-known kindness, sent me the above proof, unsolicited, and his concluding remarks were : " I think the formula $K=4\rho Vsa/3I$ is only likely to be wrong by some factor like π, or $\sqrt{3}$, or $\sqrt{2}$, or some combination thereof. If $KT/4$ is of anything like the order 1/10,000, I doubt whether the air damping can be disentangled from the fatigue of the suspending fibre." In a subsequent letter Prof. Poynting wrote that he had just found that something like a similar solution had been already obtained by Sir G. G. Stokes. Prof. Poynting made it clear that his proof was only approximate, and that he did not see how to work the problem accurately. In thanking Prof. Poynting, I said that if I published this solution I should, of course, acknowledge the authorship, so I take it that he would not have objected to my mentioning his name as I have done, in connection with this proof.

Before proceeding to apply the formula, it should be remarked that the numerical value of $KT/4$ in the following cases is much greater than 1/10,000, and so the criterion above is satisfied.

It is seen that from the above expressions

$$\rho=3I\lambda/VsaT,$$

$$\therefore \text{ the vacuum pressure, } P \text{ (in mm.)}=\frac{760}{1{\cdot}28\times10^{-3}}\cdot\rho.$$

In these experiments V (*i.e.*, velocity of mean square)

$$=5\times10^4, \text{ for oxygen at } 17^\circ\text{C}.$$

$$\therefore\ P=35{\cdot}6\cdot\frac{I}{saT}\cdot\lambda.$$

In any given experiment I/saT is constant. But with lapse of time, as the vacuum deteriorates after the sealing off, λ will increase, so, for convenience, we have :—

$$P=(35{\cdot}6)\,.\,C\,.\,\lambda,$$

C being a constant.

To quote an example : Readings were taken over a period of five weeks in 1914, the data are :—

$a=1{\cdot}6$ cm.
$I=7{\cdot}2\times(1{\cdot}6)^2$ gm. cm.^{2}.
$s=145\times0{\cdot}07$ cm. (Silver chains were used for the masses carried by the torsion beam, the wire diameter being 0·7 mm.)
$T=132$ sec.
log. $C=\bar{2}{\cdot}0343$.

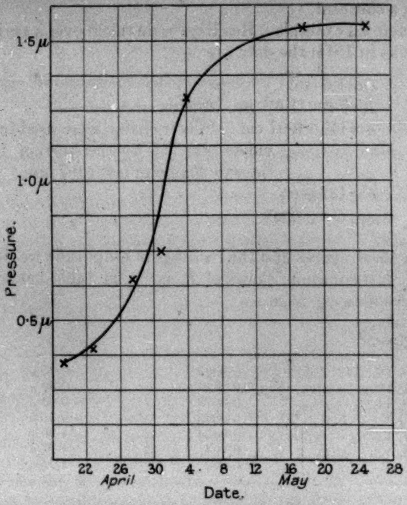
From these figures and the readings of amplitude we derive, from the last equation, values of P, as in the table below and curve on following page :—

Date.	λ.	P.
19, IV., 14	0·0009	0·35 μ
23, IV.	0·0011	0·40 μ
28, IV.	0·0016	0·65 μ
1, V.	0·0023	0·75 μ
4, V.	0·0034	1·30 μ
18, V.	0·004	1·55 μ
25, V.	0·004	1·55 μ

At the end of the five weeks considerable damping occurred in the torsion beam oscillations, so that the tube was opened and re-exhausted, and a new series of readings taken. The vacuum for which these results are given had been prepared during four days of heating and pumping, a total of 55 hours' exhausting, during which the whole system, except the Gaede pump, had been kept at over 100 deg. C. A large carbon tube, included in the vacuum, had been at 350 deg. C. for the whole time. At the end of evacuation the tube was well washed out with oxygen. It was then sealed off and allowed to cool. The carbon, let down to room temperature and kept connected to the vacuum, would then absorb residual gases.

1 μ of mercury is used throughout as the unit of pressure for high vacua, as being more convenient than a lengthy decimal fraction of 1 mm. of mercury.

The relation of gas pressure to damping in a vacuum has been investigated by J. L. Hogg (Amer. Acad. "Proc." XLV., pp. 3-7, 1909), who obtained an empirical expression from observations of both pressure and damping.



This method could not readily be applied in my experiments, for it would involve extensive subsidiary experiments.

ABSTRACT.

In experiments on the Newtonian Constant ("Phil. Trans.," May, 1916) the author used a torsion balance in a vacuum which varied in different cases from 15 mm. to 0·00001 mm. pressure. Before sealing the vessel the pressure was determined by a McLeod gauge. Values of the pressure after sealing off were deduced, in the case of the higher vacua, from observations of the damping of the torsion system.

The formula employed is due to the late Prof. Poynting, and can be expressed in the form—

$$P=35{\cdot}6\frac{I}{sa^2T}\lambda,$$

where I=moment of inertia of suspended system, s=area of surface (supposed plane) which is experiencing the resistance, a=mean

distance of plane from centre of rotation, T=period of oscillation, and λ=the observed logarithmic decrement.

A table and curve are given showing the relation between P and λ.

DISCUSSION.

Mr. F. J. W. **Whipple** (communicated remarks): Dr. Shaw's quotation from Prof. Poynting's letter is in the nature of an invitation to a search for missing factors in the formula for the resistance to the motion of a solid in a rarefied gas. It appears to me that there are two reasons for the introduction of such factors. In the first place, the mean velocity of the particles of the gas is not the same as the root-mean-square velocity. If Prof. Poynting's argument is modified to allow for this distinction, the velocities being supposed to be distributed according to Maxwell's law, the formula in question becomes, I believe,

$$K=4\sqrt{\frac{2}{3\pi}}V sa\rho/I.$$

Some further modification in the formula would seem to be necessary if it is to be applied to the motion of a cylinder. The particles will rebound from a cylinder and a plane lamina in quite different ways, and it is not to be supposed that the resultant force on a cylinder is equal to that which would be experienced by a lamina occupying the central section. May I be allowed to add a word in favour of the use of absolute units for the measurement of pressure. In the theoretical part of this Paper the pressure is given in absolute measure—*i.e.*, if the C.G.S. system is used, in dynes per square centimetre; but when the results of experiment are under discussion there is a transition to micrometers of mercury. For meterological work the millibar, 1,000 dynes per square centimetre, is now in general use. As it happens that the pressure due to a millimetre of mercury under the standard conditions—*i.e.*, at the fusing point of water at sea level in latitude 45° is very nearly four-thirds of a millibar—the change of units is easily effected. The pressures shown in Dr. Shaw's table run from 0·47 to 2·07 dynes per square centimetre or microbars.

Dr. **Shaw** communicated the following reply: The value of K, according to Mr. Whipple, should be raised by multiplying it by about $\sqrt{2}$. This possibility was foreseen by Prof. Poynting, and is mentioned in the text of the Paper. The forces due to molecular rebound on (*a*) a plane lamina area A, and on (*b*) a cylinder whose aerial section has area A, are not identical, but I imagine that (*a*) and (*b*) would give results differing by less than, say, 20 per cent., so that for this approximate solution we need not discriminate between the two cases. Any plea for uniformity in our symbols and for the C.G.S. system is well-nigh irresistible, especially where, as here, the results are not accurate enough to preclude the use of the simple multiplier 4/3. Still, an experimentalist measures pressures in millimetres of mercury both on barometer and McLeod gauge. For exact conversion to C.G.S. units one remembers that 1 atmos. pressure is not accurately 10^6 dynes/cm.^{2}, and that 1000/760 is not accurately 4/3; and there is some excuse for the experimentalist if he forbears to do the troublesome conversion from the measuring units to the ideal units, and leaves that process to the philosopher on those rare occasions when the conversion is required.

XII. *A Diffraction Colour Box.* *By* ARTHUR W. CLAYDEN, *M.A., Principal of the Royal Albert Memorial University College, Exeter.*

RECEIVED JANUARY 11, 1917.

THE apparatus described in the following pages was constructed with the object of ascertaining whether the known phenomena of colour vision were based upon any numerical relations between the oscillation periods of the waves of light concerned; and, if so, of endeavouring to determine those numerical relations.

For the purpose in view it was necessary to have some form of apparatus which would yield pure rays whose vibration periods or whose wave-lengths could be easily read with sufficient accuracy, and which would admit of two or three rays of known periods being easily superposed and combined.

Neither Maxwell's colour box nor Abney's colour patch apparatus seemed quite suitable, and after numerous preliminary experiments with various arrangements a device was adopted which meets all requirements, and has the advantage of very great simplicity of manipulation.

It is a combination of the principle of the concave grating with Maxwell's colour box.

A silver on glass speculum by Calver, 6 in. in diameter, and 4 ft. focus, was removed from a telescope, and supported in a vertical position on adjusting screws, as at M in the diagram.

An adjustable slit was supported at S at the centre of curvature of the mirror, so that the image of the slit should be formed on the slit itself.

A Thorpe grating of 14,475 lines to the inch on parallel glass was placed in front of the mirror, so that the grating pressed gently against the silver surface of the mirror, and the rest of the mirror was covered with a diaphragm of black card.

This arrangement is found to give well defined spectra on both sides of the central image in sharp focus along the circumference of a circle, of which the slit and mirror form the ends of a diameter.

The position of the grating was carefully adjusted, so that the spectra were spread out in a horizontal plane.

The spectrum of the first order on one side was selected for use.

The accompanying diagram explains the construction of the apparatus.

S is the slit and G the grating, each carried on a stand which permits the necessary adjustments for exact focus and parallelism between the slit and the lines of the grating.

After getting accurate focus the distance G S was divided equally at P.

A wooden arc at A A′ shows the position of the spectrum.

Two equal arms P B and P B′ are pivoted at P, and carry sliding carriages C and C′, which move smoothly on the arc A A′.

Brass pivots are fixed to each sliding carriage, so that the axis of the pivot coincides exactly with the arc on which the spectrum is in focus.

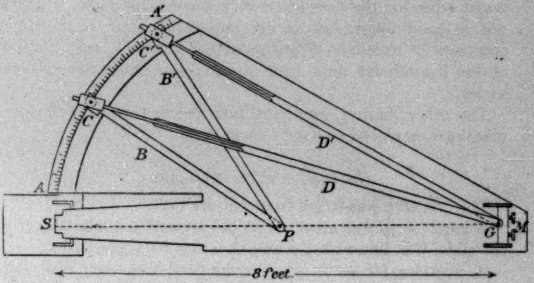


Diagram of the Apparatus.

A telescope eye-piece attached to the sliding carriage and focussed for one part of the spectrum would be in focus for all parts, but in order to see the spectrum properly it would need to be directed towards the grating. This is automatically effected, thus :—

Two long light rods D D′, are pivoted on an axis immediately below the grating, and are attached at their other ends by tubes sliding smoothly to upper portions of the carriages C C′, so constructed as to swing steadily on the brass pivots.

The eye pieces are attached to these upper portions.

The action of the apparatus is as follows : The arms P B, P B′, constrain the sliding carriers to slide on the correct arc, and the rods D D′ oblige the eye-pieces to turn always to the grating, so that if any part of the spectrum is brought to focus in either eye-piece, any other part of the spectrum may be observed by simply sliding the carriage from point to point.

The whole apparatus is encased in a light proof covering (except where the eye-pieces move) made of a wooden framework covered with thick brown paper lined with black cotton cloth. Pieces of the same black cloth are suspended inside the framework to act as numerous diaphragms, and the moving parts are separated from the path of the rays by a division of black cloth.

Sliding black curtains are provided on each side of, and between, the sliding carriages, so that practically no extraneous light can disturb the results.

The use of the Thorpe replica on parallel glass is probably responsible for the formation of a second spectum, which is not in good focus, but in my apparatus there is an upper spectrum of excellent definition showing a very large number of the Fraunhofer lines from the A group up to the extreme violet.

The other, fainter, spectrum falls below this, and the eye-pieces are so placed as not to see it at all.

Graduation of the Apparatus.

The first step was to graduate the arc C C′.

A paper strip was attached to the arc and a mark affixed to the slider C in line with the axis on which its upper part moved.

A considerable series of spectral lines was then observed, taking lines of known wave-lengths from the flame spectra of sodium, lithium, strontium, indium and potassium, the vacuum tube spectra of hydrogen and mercury, and the spark spectra of magnesium and zinc.

The position of the mark on the carrier when each line was central in the field of view was marked on the paper, and found to be proportional to the wave-length.

The dispersion of the apparatus may be gathered from the facts that the sodium D lines are about $\frac{1}{16}$ in. apart; there is approximately 1 in. between the red lines of lithium and hydrogen, and the whole spectrum is about 22 in. long.

Having made a series of observations with one carrier the second was brought into use and centred on the hydrogen red line. A mark was then made on the carrier coincident with the mark on the scale, and on moving the carrier it was found to read the same as the first in all parts of the spectrum.

Transformer.
Observing Telescope directed to slit, to the left of which is the sliding contact.
Contact keys.
Additional Arc to carry third lamp.
Graduated Arc.

The Apparatus as it stands in my study. When in use the curtain is dropped over the whole front to cover the sliding carriers, eye-pieces and lamps ; and the window on the right of the picture is closed with shutters.

[*To face page* 178.

To study the phenomena presented by the mixture of two or more colours the following method is adopted :—

If a hydrogen vacuum tube is placed before the slit and an induction discharge passed through it, one eye-piece can be so placed as to see the red line and the second one of the other lines, say, the blue.

If, now, an incandescent filament giving white light is placed in the same position in the eye-piece as the image of the red facts that the sodium D lines are about $\frac{1}{10}$ in. apart; there is approximately 1 in. between the red lines of lithium and hydrogen, and the whole spectrum is about 22 in. long.

Having made a series of observations with one carrier the second was brought into use and centred on the hydrogen red line. A mark was then made on the carrier coincident with the mark on the scale, and on moving the carrier it was found to read the same as the first in all parts of the spectrum.

As it seemed desirable to deal rather with vibration frequencies than with wave-lengths the next step was to prepare a second scale giving such values.

The oscillation frequencies employed in recent spectroscopic work are inconveniently large numbers, and give the number of waves in a given space. It seemed preferable to deal with the number of oscillations in a given time.

The number of oscillations per second was then computed for a number of different wave-lengths taken at intervals along the spectrum, and the results were plotted in the form of a curve from which the periodicity of any wave-length could be read.

The number of oscillations per second varies from about 39×10^{13} in the extreme red up to about 76×10^{13} at the other end of the visible spectrum.

Observation soon showed that the smallest change of wave-length that could be recognised by the eye as a change of colour was greater than that which corresponded to a change of period of 10^{12} vibrations per second, or to a change of one vibration more or less in $1/10^{12}$ second.

If we take this as our unit of time we get the vibration frequencies of visible light expressed by a series of three figure numbers easily comparable with those expressing the vibrations of sound.

The unit $1/10^{12}$ second might be called a twelfth-second after the analogy of the tenth-metre.

Having the curve showing wave-lengths in terms of vibration periodicity per twelfth-second another scale was constructed and fitted to the wave-length scale on the graduated arc of the apparatus.

The accuracy of the graduation was tested repeatedly and minor readjustments made until it was found to read correctly to within the experimental limits of one vibration per twelfth-second.

Mixtures of Colours.

To study the phenomena presented by the mixture of two or more colours the following method is adopted :—

If a hydrogen vacuum tube is placed before the slit and an induction discharge passed through it, one eye-piece can be so placed as to see the red line and the second one of the other lines, say, the blue.

If, now, an incandescent filament giving white light is placed in the same position in the eye-piece as the image of the red line, red light of exactly that same period will be returned to the slit. If, then, the slit is opened widely and a small observing telescope is focussed on the grating it will be seen to be brightly illuminated by light of that wave-length only.

Similarly with the blue line, or any other line; and, if both white filaments are incandescent at the same time, the telescope will show the grating illuminated by light of those two periodicities, and none other than those two.

For this purpose some small metallic filament lamps such as are used for motor work are mounted so that they can be slipped into the eye-pieces in such a position that the filament, is in the position of the image of a spectral line. The filament is a small Osram spiral whose width is less than half the distance between the sodium D lines.

In the earlier experiments the current for the lamps was drawn from batteries, but subsequently these were replaced by a low voltage current obtained by transforming the domestic supply down to 8 and 5 volts.

The two lamps are connected with the two branches of a divided circuit, one end of which is a sliding contact moving upon a coil of fine iron wire wound on a long bone knitting needle. By altering the position of the contact the relative resistances of the two arms of the circuit can be varied and the intrinsic brilliancy of the lamps easily adjusted at pleasure.

Each lamp circuit also includes a spring tapping key.

A board is fixed under the slit of the apparatus, and this carries the observing telescope, the keys and the sliding contact.

Meanwhile the nearer of the two sliding carriers is easily reached and moved by the left hand.

Hence the lamps can be turned on and off, their relative brilliancy changed, and the colour of the light received from one of them modified, without removing the eye from the observing telescope.

The purity of the light returned to the observing telescope was repeatedly tested by substituting an ordinary spectroscope. The analysis of the light always gave clear, sharply defined lines without any trace of any light other than that of the periodicities indicated by the position of the carriers on the scale.

As it is necessary for some purposes to have a third lamp, one is mounted so that it can be placed by hand in any desired position on the arc, but its support is not provided with the automatic focussing and directing rods.

ABSTRACT.

The apparatus consists essentially of a very simple concave grating spectroscope, of which the slit and grating are situated at opposite diameters of a circle, the spectrum being formed on the arc of this circle. Two independent arms carry fittings on which may be placed either telescope eyepieces or small electric lamps. With the slit of the instrument illuminated by a suitable source, the eyepieces can be set so that any two desired wave-lengths are in the centres of their respective fields of view. The eyepieces are then replaced by the small lamps (the filaments coinciding with the previous positions of the crosslines), and the grating is observed with a small telescope pointed towards the widened slit: the whole of its surface is seen to be illuminated with a mixture of the two colours on which the eyepieces were originally set.

The "concave grating" employed consists of a Thorpe replica of a Rowland plane grating of 14,475 lines to the inch, mounted with its ruled surface in contact with the surface of a concave mirror of 4 ft. focal length. This forms an admirable substitute for the more expensive concave grating.

The author prefers to state results in terms of the number of oscillations per unit of time.

Observations showed that the smallest change of wave-length which could be recognised by the eye as a change of colour was greater than that which corresponded to a change of period of 10^{12} vibrations per second, or to a change of one vibration more or less in $1/10^{12}$ second.

DISCUSSION.

Mr. Trotter asked how the minimum difference of wave-length detectable as colour difference compared with the values obtained by Dr. Edridge-Green.

Mr. T. Smith said that experiments on the minimum colour difference which the eye could detect were valueless unless the exact conditions under which the experiments were made were stated. For example, the comparison of two colours side by side is materially affected by the presence of a black space between them. In Dr. Edridge-Green's experiments the conditions were not good. The experiments were made somewhat as follows:—While viewing the extreme red end of the spectrum, a shutter is brought down towards the red so as to cut off everything above the region to which the observer would give one distinguishing name, such as deep red. A second shutter was then brought up from the red end to cut off the first patch, and the first shutter moved along the spectrum until the boundary of the next "monochromatic" patch was located. By this means the whole spectrum was divided up into about 25 monochromatic patches. It is not difficult to repeat these observations, but if they are compared with measurements made in another way entirely different results are obtained. Lord Rayleigh was able, for instance, to distinguish between the colour of the two sodium lines. He had two spectra, one above the other, the upper one being inverted, so that D_1 and D_2 of one spectrum coincided with D_2 and D_1 of the other. Colour comparisons should always be made from simultaneous observations of the two colours.

Prof. C. V. Boys said he would like to emphasise Mr. Smith's remarks about the effect of varying conditions on the visual estimates of colour. In the colours of soap films, for instance, this difficulty is met with.

Where the thickness is nearly zero the colour is black. As the thickness increases the colours of the first order are successively encountered. Between the blue and yellow there is a tint which can only be described as a bad white. He had long been puzzled as to what this colour was, but had eventually discovered a method of overcoming the deceptive influence of the adjacent colours. Usually these films are traversed by long thin folds. The thickness of the folded film is everywhere three times that of the adjacent unfolded film. Points can be found where this doubtful region between the blue and yellow, occurring on the triple film, can be directly contrasted with the first order white in the adjacent single film. It is then seen to be a very good green. A point which had surprised him in connection with Mr. Clayden's apparatus was that the plane grating, which could not be in actual contact with the mirror except at its edges, nevertheless behaved as if it were in good optical contact.

Dr. W. **Eccles** pointed out that the minimum colour difference estimated by Mr. Clayden to be perceptible to the eye, if expressed in the same way as an interval in music, works out as about 1·0013. The tone difference just perceptible to the trained musical ear has been stated to be one seven-hundredth of an octave, which works out as 1·001. It is interesting to observe that the sensitiveness of the eye to pitch differences is, like its sensitiveness to inflow of energy, of the same order as that of the ear.

Mr. **Clayden**, in reply, said that in making his experiments on the minimum difference of wave-length required to produce a noticeable change of colour he had arranged matters so that by using a reversing or non-reversing eyepiece in the observing telescope one or other colour was uppermost in the field of view. The difference in wave-length was then reduced until it was impossible to say which was which. By this method he could detect a difference of two vibrations per $1/10^{12}$ sec., but at the red and violet ends of the spectrum the detectable difference greatly increased. With regard to the President's last remarks, he had himself been very agreeably surprised to find that the performance of the grating was so satisfactory.

An Arrangement for Showing, by Projection on a Screen, the Coloured Fringes Produced when a Transparent Grating is Laid with its Ruled Face against a Plane Mirror, was shown at the Meeting on March 9th, by Mr. A. W. CLAYDEN.

THE light from a small lantern is incident in an approximately parallel beam on the interface of the grating and mirror, which are held together in a wooden framework supported on a stand. The position of the grating-mirror combination is adjusted so that the central reflected beam, or any one of the diffracted beams, is returned just alongside the lantern. The beam passes through a projection lens which focuses the fringes on a distant screen. The fringes thus obtained are very bright, and the simplest apparatus and adjustments suffice for their production.

DISCUSSION.

Dr. R. S. WILLOWS mentioned that Fellows had informed him that these fringes had been described by Barus in the "Philosophical Magazine" (July, 1910, and July, 1911). He understood that they had also been exhibited by Rheinberg at the Optical Convention in 1910. Mr. Clayden's method of exhibiting them by projection would, he thought, be distinctly useful to teachers.

An Apparatus for Studying the Effect of Hertzian Waves on the Beating of the Heart was Exhibited at the Meeting on March 9th by Prof. W. M. COLEMAN.

A SIMPLE pendulum, consisting of a cylindrical brass bob terminating in a pointed wire coaxial with the bob, hangs by a piece of string above one of the terminals of an induction coil, so that in its lowest position the point of the bob is within sparking distance of the terminal and vertically above it. The bob is connected by a piece of flexible wire to the other terminal of the coil. When the pendulum is set oscillating there is a shower of sparks every time the bob passes its lowest position The frequency of intermittence can be varied by altering the length of the suspension. By adjusting the period of the pendulum nearly to the time of a heart-beat any possible effect on the rate of the beating may be observed. Prof. Coleman stated it to be his experience that the heart beats tend to acquire the same rate as the sparking.

The condensed discharge from two Leyden jars is employed.

XIII. *A Note on the Derivation of the General Equation for Wave Motion in an Elastic Medium.* By J. A. Fleming, *M.A., D.Sc., F.R.S.*

Received March 1, 1917.

In order that a wave may be propagated in a medium the latter must possess the power of storing energy in each element of volume in two ways—viz., as potential energy or energy of strain in its most general sense, due to some type of displacement made against an elastic resistance; and also as kinetic energy in virtue of the possession of some quality comparable with density, such that change of displacement bestows on the element energy, whilst the strain is changing. The elasticity of the medium is measured by the ratio of stress to strain involved in the displacement. A wave consists in a periodic transformation of the energy stored per element of volume of the medium from kinetic to potential and vice versa, in such fashion that at points lying on a line called the direction of wave propagation the energy continuously varies in form from point to point; being wholly kinetic or wholly potential at the same moment at certain equidistant points at intervals called half a wave length; whilst at intermediate points it is partly in one form and partly in the other. The characteristic of wave motion is therefore that a certain physical state in the medium, which may generally be called a strain, varies from point to point in a cyclical manner along certain lines at the same instant, and also varies at the same point in a cyclical manner from instant to instant.

If, therefore, we choose two closely adjacent points in the line of propagation it is possible to say that at the same instant the difference in the states of the strain at these points is precisely the same as the difference between the states at one of these points at some two successive closely adjacent intervals of time.

This strain may be regarded as produced by a corresponding type of stress, and since the quotient of stress by strain is a measure of the elasticity concerned, we can say that the product of strain and elasticity is a measure of the stress producing that strain. The stress may also be measured in another manner as the product of the mass per unit volume—or density of the medium—or equivalent constant and the strain acceleration or rate of change of the displacement or strain. If we then express in mathematical language this fact, we arrive at a differential equation the solution of which gives us all possible information about the wave motion.

Let us take first the simple case of wave propagation in an elastic material contained in a rigid tube, no friction existing between the material contained in the tube and its walls. Let the tube be of uniform cross-section of unit area. Let m be the density of the medium or kinetic energy factor, and let e be the elasticity or potential energy factor. Let φ denote some form of strain in the material which is propagated through it, whether longitudinal or lateral displacement, torsion, shear or compression. Let $\varphi=d\varphi/dt$ denote the *strain-velocity* or time rate of change of displacement, and $\ddot{\varphi}=d^2\varphi/dt^2$ denote the *strain acceleration* or rate of change of displacement at the same point. Let r be the distance along the tube axis measured from some origin, and let two parallel right sections be taken at distances r and $r+\delta r$.

Let φ be the strain in the cross section at distance r and $\varphi+\delta\varphi$ at $r+\delta r$.

Then the proper measure of the wave-making strain at the distance r is the limiting value of the ratio $\delta\varphi/\delta r$—viz., $d\varphi/dr$. This is clear from the analogy with a bar of length l, increased in length by δl by any applied stress, where $\delta L/l$ is called the longitudinal strain; and this last multiplied by Young's modulus is the stress causing that strain.

Hence, if e is the elastic coefficient, the *wave-making stress* or effective stress at the same point will be $ed\varphi/dr$, and the quantity

$$\frac{d}{dr}\left(e\frac{d\varphi}{dr}\right)\delta r$$

will represent the difference between the wave-making stress at r and that at $r+dr$. This change in the stress may be also expressed by the product of the mass of the intermediate section and the strain acceleration—viz., by $m\delta r\ddot{\varphi}$.

Accordingly, if e is independent of r, we have the equation—

$$m\frac{d^2\varphi}{dt^2}=e\frac{d^2\varphi}{dr^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$\frac{d^2\varphi}{dt^2}=\frac{e}{m}\frac{d^2\varphi}{dr^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 1A)$$

Which is the well-known differential equation for such plane waves.

The solution of the equation (1) is :—

$$\phi=f_1\left(r-\sqrt{\frac{e}{m}}t\right)+f_2\left(r+\sqrt{\frac{e}{m}}t\right), \quad . \quad . \quad . \quad (2)$$

as can easily be found by trial.

This equation indicates a wave moving with velocity $\sqrt{e}/\sqrt{m}$, having a wave form whose profile has the equation $\phi=f(r)$, or a certain function of r.

For $\phi=f(r)$ denotes a certain curve, of which the abscissa is r and ordinate ϕ, the curve being at rest. But $\phi=f\left(r-\sqrt{\frac{e}{m}}t\right)$ denotes the same curve moving bodily forward without change of form with a velocity $\sqrt{e}/\sqrt{m}$, since ϕ remains the same in the last expression if for r we put $r+r'$, and for t we put $t+t'$, provided that $r'/t'=\sqrt{e}/\sqrt{m}$.

In other words the ordinate of magnitude ϕ arrives at a point at a distance r' further on at a time later by t'.

Accordingly a disturbance of any kind made in an elastic-dense non-viscous medium filling a rigid tube is propagated with a velocity $\sqrt{e}/\sqrt{m}$, and without change of wave-form, if we assume the absence of friction between the elastic medium and the walls of the tube.

We can next derive in the same simple manner the differential equation for wave motion in a homogeneous, infinitely extended, elastic, dense medium due to a disturbance made at some point taken as origin, if we assume that the elasticity e of the medium and its density m are constant and independent of the distance from the origin.

Describe round the origin two concentric spheres of radii r and $r+\delta r$ respectively. As before, let ϕ be the value of the strain at distance r and $\phi+\delta\phi$, that at $r+\delta r$ at the same instant. Then the wave-making strain at all points of the sphere of radius r is $d\phi/dr$, and the wave-making stress is $ed\phi/dr$. This last expression is, however, the stress per unit area, and hence the total stress for the whole spherical surface of radius r is $4\pi r^2 e\frac{d\phi}{dr}=4\pi er^2\frac{d\phi}{dr}$. The increment in this in passing from radius r to radius $r+\delta r$ is $4\pi e\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right)\delta r$.

The mass of the spherical shell of thickness δr is $4\pi r^2 m\delta r$, where m is the density and hence, equating the two expressions for the total stress, we have :—

$$\frac{d^2\varphi}{dt^2}m4\pi r^2\delta r=4\pi e\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right)\delta r, \qquad (3)$$

or

$$m\frac{d^2\varphi}{dt^2}=\frac{e}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right), \qquad (4)$$

or, if e is independent of r :—

$$m\frac{d^2\varphi}{dt^2}=e\left(\frac{d^2\varphi}{dr^2}+\frac{2}{r}\frac{d\varphi}{dr}\right). \qquad (5)$$

If the rectangular co-ordinates of the extremity of the radius r are x, y and z, so that,

$$r=x^2+y^2+z^2,$$

then it is easy to show that if φ is independent of angular direction,

$$\frac{d^2\varphi}{dr^2}+\frac{2}{r}\frac{d\varphi}{dr}=\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz}. \qquad (6)$$

and, therefore, we have from (5) and (6)—

$$\frac{d^2\varphi}{dt^2}=\frac{e}{m}\left(\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz}\right). \qquad (7)$$

$$=\frac{e}{m}\Delta^2\varphi$$

and this is the well-known wave equation for space waves.

The solution of (5) and, therefore, of (7) is easily obtained as follows :—

We note that

$$\frac{d^2\varphi}{dr^2}+\frac{2}{r}\frac{d\varphi}{dr}=\frac{1}{r}\frac{d^2(r\varphi)}{dr^2}, \qquad (8)$$

as can be found by differentiation. Hence, since r is independent of t, we can write (5) in the form

$$\frac{d^2(r\varphi)}{dt^2}=\frac{e}{m}\frac{d^2(r\varphi)}{dr^2}. \qquad (9)$$

But the solution of this is

$$r\varphi=f_1(r-ct)+f_2(r+ct), \qquad (10)$$

where $c=\sqrt{e}/\sqrt{m}$.

Accordingly the solution of (5) and (7) is

$$\varphi=\frac{1}{r}F_1(r-ct)+\frac{1}{r}F_2(r+ct), \quad . \quad . \quad . \quad . \quad (11)$$

where f and F stand for some function of the quantity $r\mp ct$.

This shows that the waves travelling outward from the source attenuate in amplitude inversely as the distance, but travel without change in wave-form, with velocity $c=\sqrt{e}/\sqrt{m}$. In the above case we are dealing with a single spherical wave originating in a point source.

On showing this method of arriving at equation (7) to Prof. J. W. Nicholson he suggested to the author that the method could be adapted to the case of waves generally, not limited to the above simple point source instance, by the application of Huyghen's principle. The transformation given in equation (6) above, viz. :—

$$\frac{d^2\varphi}{dr^2}+\frac{2}{r}\frac{d\varphi}{dr}=\Delta^2\varphi$$

is, of course, only true when φ is a function of r and not of angles or orientations. The Laplacean operator

$$\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}=\Delta^2$$

is, however, an invariant with regard to origin—that is, it retains the same mathematical form if for x, y and z we substitute $x+\alpha$, $y+\beta$, $z+\gamma$, where α, β, γ are constants. Hence it is independent of a shift of origin.

Huyghen's principle states that when any set of waves affect a point in a medium they cause that point to send out its own spherical disturbance, and the waves from each point therefore satisfy equation (7) with that point as origin. But since the operator Δ^2 is independent of origin in form, this means that the equation is satisfied for every origin. The strains also are additive by the principle of superposition of small motions, and hence the equation

$$\ddot{\varphi}=c^2\Delta^2\varphi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

is satisfied generally for the whole assembly of waves or for the general wave motion at any place and time, no matter how or where originating. Hence, it is true generally.

Finally, we can apply the same method of building up the differential equation to the case of electromagnetic waves in a pure dielectric and arrive at the familiar Maxwell equation for this type of wave. In this case we consider that the magnetic permeability μ of the dielectric is the kinetic energy factor corresponding to the density of the medium, and that the reciprocal of the dielectric coefficient K is the elasticity coefficient of the medium. For if D is the dielectric displacement or strain and E the electric force or stress at any point in the dielectric then on a rational system of units $D=KE$ or $E=\frac{1}{K}D$. As above, the wave-making strain is dD/dr, if we assume that the waves originate in a point source at distance r. The wave-making stress is, therefore, $\frac{1}{K}\frac{dD}{dr}$ per unit of surface perpendicular to the direction of r.

The strain-acceleration is $\ddot{D}=\frac{d\,D}{dt^2}$. Hence, for a spherical shell of the medium of radius r and thickness δr, we have the equation as before—

$$4\pi r^2\delta r\mu\frac{d^2D}{dt^2}=\frac{4\pi}{K}\frac{d}{dr}\left(r^2\frac{dD}{dr}\right)\delta r, \quad . \quad . \quad . \quad (13)$$

or

$$\frac{d^2D}{dt^2}=\frac{1}{\mu K}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dD}{dr}\right)$$

$$=\frac{1}{\mu K}\left(\frac{d^2D}{dr^2}+\frac{2}{r}\frac{dD}{dr}\right)$$

or

$$\frac{d^2D}{dt^2}=\frac{1}{\mu K}\Delta^2 D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

This assumes that D is a function of r only, and that μ and K are constants. It follows that such spherical electromagnetic wave travels outwards with velocity $1/\sqrt{\mu K}$, and that the amplitude varies inversely as the distance from the origin.

The equation (14) is usually obtained directly from the Faraday law of electromagnetic induction, and the Maxwell-Ampère law by the Maxwell's applications to a pure dielectric as follows :—

By Faraday's law we have—

$$-\mu\dot{H}=\text{Curl } E, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where H is magnetic force and E electric force at any point in the dielectric.

By the Maxwell-Ampere law—

$$K\dot{E}=\text{Curl } H, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

but if D is displacement or electric strain then in rational units $D=KE$. Hence, eliminating H, $\dot{H}$, E, $\dot{E}$ from the last three equations we have—

$$\mu K\ddot{D}=-\text{Curl}^2 D. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

But E, D and H are here non-divergent vectors, as we assume no electrified bodies or magnetic poles are in the field.

For such vectors it is easy to show that the operator $-\text{Curl}^2\,(\quad)$ is identical with the Laplacean operator $\Delta^2\,(\quad)$ (*see* "The Wireless Telegraphists' Pocket Book," J. A. Fleming, p. 45, for simple proof of this).

Hence, (17) becomes in this case—

$$\ddot{D}=\frac{1}{\mu K}\Delta^2 D, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is the same equation as (14).

The solution is—

$$D=\frac{1}{r}F(r\mp ct), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $c=1/\sqrt{\mu K}$, and F is some function.

The above method of arriving at the well-known differential equation (7) for wave motion will probably be found by some students of physics more easy to follow than the methods based on the hydrodynamic equations; whilst it is more general than the method given in most books on acoustics, in which the displacement is limited to the form of a wave of condensation and rarefaction in a gaseous medium. It is also equally applicable as shown to the case of electromagnetic waves.

Added April 30th.

The above treatment assumes an isotropic medium for the wave propagation, but it is easy to modify it for the case of anisotropic media. Consider the case of electromagnetic waves in a dielectric with principal dielectric coefficients K_1, K_2, K_3 in the three axial directions and constant permeability μ. Then, if D_1, D_2, D_3 are the electric displacements in the axial directions corresponding to any general displacement D, and if $(\text{Curl}\,^2 D)_x$, $(\text{Curl}\,^2 D)_y$, $(\text{Curl}\,^2 D)_z$ denote the axial components of the vector Curl ^2D or Curl (Curl D), we can write the three equations true for the axial directions as follows :—

$$\mu K_1\ddot{D}_1=-(\text{Curl}\,^2 D)_x,$$
$$\mu K_2\ddot{D}_2=-(\text{Curl}\,^2 D)_y,$$
$$\mu K_3\ddot{D}_3=-(\text{Curl}\,^2 D)_z.$$

But the axial components of the vector Curl D are respectively—

$$\left(\frac{dD_3}{dy}-\frac{dD_2}{dz}\right),\quad \left(\frac{dD_1}{dz}-\frac{dD_3}{dx}\right),\ \text{and}\ \left(\frac{dD_2}{dx}-\frac{dD_1}{dy}\right),$$

and the axial components of the vector Curl ${}^2D$ are, therefore,

$$\frac{d}{dy}\left(\frac{dD_2}{dx}-\frac{dD_1}{dy}\right)-\frac{d}{dz}\left(\frac{dD_1}{dz}-\frac{dD_3}{dx}\right),$$
$$\frac{d}{dz}\left(\frac{dD_3}{dy}-\frac{dD_2}{dz}\right)-\frac{d}{dx}\left(\frac{dD_2}{dx}-\frac{dD_1}{dy}\right),$$
$$\frac{d}{dx}\left(\frac{dD_1}{dz}-\frac{dD_3}{dx}\right)-\frac{d}{dy}\left(\frac{dD_3}{dy}-\frac{dD_2}{dz}\right).$$

Equating these last expressions to $-\mu K_1\ddot{D}_1$, $-\mu K_2\ddot{D}_2$, $-\mu K_3\ddot{D}_3$ respectively, we have the equations—

$$\mu K_1\frac{d^2D_1}{dt^2}=\frac{d^2D_1}{dy^2}+\frac{d^2D_1}{dz^2}-\frac{d^2D_2}{dxdy}-\frac{d^2D_3}{dxdz},$$
$$\mu K_2\frac{d^2D_2}{dt^2}=\frac{d^2D_2}{dx^2}+\frac{d^2D_2}{dz^2}-\frac{d^2D_1}{dxdy}-\frac{d^2D_3}{dydz},$$
$$\mu K_3\frac{d^2D_3}{dt^2}=\frac{d^2D_3}{dx^2}+\frac{d^2D_3}{dy^2}-\frac{d^2D_1}{dxdz}-\frac{d^2D_2}{dydz}.$$

But these equations are identical with those given by Maxwell (*see* " Electricity and Magnetism," Vol. II., second edition, § 794), which he shows lead to Fresnel's expressions for wave propagation in a non-isotropic medium, provided we consider the electrostatic potential to be zero.

ABSTRACT.

The Paper explains a simple method of arriving at the general differential equation for wave motion—viz.,

$$\frac{d^2\varphi}{dt^2}=c^2\left(\frac{d^2\varphi}{dx^2}+\frac{d^2\varphi}{dy^2}+\frac{d^2\varphi}{dz^2}\right),$$

where c is the velocity of propagation of the wave. A wave motion consists in the propagation of some form of strain through an elastic dense medium or one having comparable qualities. Corresponding to this strain there is an analogous stress in the medium, and we can express this stress mathematically in two ways—viz., kinetically, as the product of the density, and the time rate of change of the strain, or by $m\frac{d^2\varphi}{dt^2}$ per unit of volume, and also statically as the space increment of the expression $e\frac{d\varphi}{dr}$, where e is a

coefficient of elasticity and r is a vector denoting direction of propagation. Hence, for plane waves we can write the equation

$$m\frac{d^2\phi}{dt^2}\delta r=\frac{d}{dr}\left(e\frac{d\phi}{dr}\right)\delta r,$$

or if m and e are independent of distance, we have

$$\frac{d^2\phi}{dt^2}=\frac{e}{m}\frac{d^2\phi}{dr^2},$$

which is the known differential equation of wave motion for plane waves. Its solution is $\phi=F\left(x\mp\sqrt{\frac{e}{m}}t\right)$, where F is some function. In the same manner for spherical waves originating in a point source we can write the stress equation

$$4\pi r^2 m\delta r\frac{d^2\phi}{dt^2}=\frac{d}{dr}\left(4\pi r^2 e\frac{d\phi}{dr}\right)\delta r,$$

which at once gives

$$\frac{d^2\phi}{dt^2}=\frac{e}{m}\left(\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right);$$

or if $r^2=x^2+y^2+z^2$, we have

$$\frac{d^2\phi}{dt^2}=\frac{e}{m}\left(\frac{d^2\phi}{dx^2}+\frac{d^2\phi}{dy^2}+\frac{d^2\phi}{dz^2}\right).$$

The solution of this last equation is

$$\phi=\frac{1}{r}F\left(x\mp\sqrt{\frac{e}{m}}\right),$$

where F is some function.

The general equation for electromagnetic wave propagation in a pure dielectric may also be obtained in the same manner. In this case the strain on the medium is the electric displacement D, and, if E is electric force, then on a system of rational units $1/K$, where K is the dielectric constant, is the electric elasticity. Hence we can consistently assume that the magnetic permeability μ corresponds to density, and then the kinetic measure of the electric stress is $\mu\frac{d^2D}{dt^2}$, and this must be equated to the space variation of the static measure of the electric stress.

For the spherical wave from a point source this leads to the equation

$$\frac{d^2D}{dt^2}=\frac{1}{\mu K}\left(\frac{d^2D}{dx^2}+\frac{d^2D}{dy^2}+\frac{d^2D}{dz^2}\right),$$

with the solution

$$D=\frac{1}{r}F\left(x\mp\frac{1}{\sqrt{\mu K}}t\right).$$

The wave is, therefore, propagated outwards without change of wave form; but the amplitude varies inversely as the distance from the source.

Since the Laplacean operator $\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}$ is an invariant, and is not altered in form by a shift of origin; and since we can by Huyghen's principle consider a wave of any kind as the resultant of a set of spherical waves originating at various point sources, we can apply the above method to the case of wave motion generally.

The method above described may be epitomised by saying that we obtain the differential equation by equating the product of strain-acceleration $\frac{d^2\phi}{dt^2}$ and density to the static measure of the stress expressed as the space variation of the product of the elasticity and the strain slope $\left(e\frac{d\phi}{dr}\right)$, which is the proper measure of the stress at the point considered.

DISCUSSION.

Mr. F. J. W. **Whipple** thought that the validity of Prof. Fleming's application of Huyghen's principle might be questioned. In the first place, he thought that the principle itself required proof just as much as the property that it was used to establish. Further, the secondary wavelet from each point in a wave front was assumed to be spherical. Was it not the case that the phase of these disturbances was a function of the direction, relative to the main wave front, as well as of the distance from the origin of the wavelet? Another interesting question was how the equations would have to be modified in the case of the energy reflected at the boundary surfaces of the medium.

Prof. C. H. **Lees** did not quite see how Prof. Fleming's treatment of the problem differed from that given in Love's "Elasticity," for example.

Prof. J. A. **Fleming**, in reply, wrote that the criticisms of the two speakers in the discussion rather missed the point of the Paper. The validity of Huyghen's principle is not in question. Most writers on physical optics have resorted to it, and he (Dr. Fleming) had never seen any explanation of the phenomena of diffraction which did not rest upon it. The proof given in the Paper that the characteristic equation proved for spherical waves with origin at the centre may be applied to waves however originating is made to rest upon the fact that the Laplacean operator is invariant with regard to origin. The object of the Paper was not to write a complete theory of wave motion, nor of elasticity, but merely to give a method, which seems valid, for arriving immediately at the characteristic equation in a manner which would appeal to students not very well versed in hydrodynamics or the theory of elasticity. Prof. Lees asks how far the treatment differs from that given, say, in Love's "Theory of Elasticity." The answer is not at all in the result; but the method used in the Paper enables us to arrive by simple considerations at the characteristic differential equation, and then to elucidate the physical meaning of its solution. It seems better for a physical and engineering student to have some approach to such conception rather than none at all. The treatment of the subject of wave motion is generally very difficult to follow in advanced text-books, whereas the practical applications of sound, water and electromagnetic waves makes some theoretical knowledge of the subject necessary.

XIV. *The Effect of Stretching on the Thermal Conductivity of Wires.* *By* A. Johnstone, *B.Sc.*

Received March 7, 1917.

Communicated by Prof. C. H. Lees, F.R.S.

Outline of Method Used.

1. Most observers, in determining the thermal conductivity of substances which conduct heat readily, have found it advisable to use the material in the form of wire and to let the heat flow along the axis. This method was adopted in the present experiments. The method is simplified if the heat flows from the centre of the wire to the two ends, which are kept at a constant temperature. In the following experiments the wire was surrounded by a vessel to ensure this, heat was supplied to the centre of the wire by passing a current of electricity through a manganin coil wound on the wire, and the temperature difference between two points on the same side of the centre was ascertained by means of two platinum coils also wound on the wire.

The Apparatus.

2. The wire to be tested $A\,A$, (Fig. 1) was fixed at each end between the jaws of screw clamps, B, joined to the brass cylinders, through which water could be circulated. To one clamp was attached a rod, T, passing through a framework, so that the tension could be applied by a nut working on a thread, T'; and to the other was attached a spring balance, K, by which the tension was measured.

The heating coil H_t was made from 14 cms. of No. 40 manganin wire closely wound on cylindrical sleeves of micanite, which could just slide on to the wire; 0·5 cm. of the wire was left unwound at each end, and to these ends were soldered covered copper leads of No. 30 gauge. The platinum thermometers (H, C, J') used were made from No. 40 single silk-covered platinum wire each of 35 cms. length, wound like the heating coil and having similar leads. This gave them a resistance of about 3 ohms at ordinary temperatures. The heating coil was placed at the centre of the wire under examination which was about 16 cms. long; one platinum thermometer was placed near to it, and the other near one of the clamps. For purposes of reference the former will be called the "hot coil" and the other the "cold coil." To support the leads from the coils against breakage, each coil was tied by silk threads to a light wooden strip which it touched at two points and to which the leads were fixed. The coils were kept at a constant distance apart by a cross strip of mica and were strengthened by a coating of shellac to prevent collapse during subsequent removal. The insulated leads from the coils were passed through

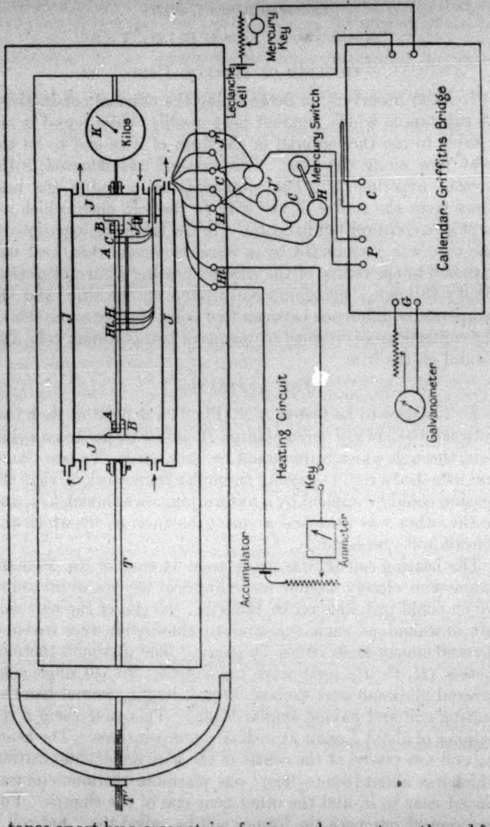

Mercury Key
Leclanché Cell
K
Kilos
Mercury Switch
Callendar-Griffiths Bridge
J'
C'
H
Ht
J'
C
H
C
P
G
J
B
A
C
J'
Ht
H
J
J
A
B
J
Heating Circuit
Key
Galvanometer
T
Accumulator
Ammeter
T'

Fig. 1.—Apparatus and Scheme of Connections.

a brass tube in one of the end clamps to mercury cups in the wood frame, and the wire and end pieces were surrounded by a water jacket. Inside and touching the jacket was a supplementary platinum coil to enable changes in the jacket temperature to be detected.

The resistances of the coils were determined by means of a Callendar-Griffiths bridge, which was read in ohms to the fourth decimal place. For the galvanometer used a deflection of 1 mm. on the scale represented a difference of 0·0002 ohm, and for the coils used this would represent a temperature difference of about 0·01°C. For the conditions of the experiments performed this would enable a difference in the conductivity of 0·05 per cent. to be detected. The current to the heating coil was supplied by an accumulator and was regulated by an adjustable resistance. The current for the bridge was supplied by a Leclanché cell in series with an adjustable resistance. Each of the platinum coils could be connected in turn with the bridge at P through the mercury key as shown in the diagram.

THEORY OF THE METHOD USED.

3. If a thin rod of uniform cross section is entirely surrounded by air in a vessel kept at constant temperature, while heat is supplied to it near one end and the other is kept in good thermal contact with the wall of the vessel, then *

$$v=\frac{Q}{(phqk)^{\frac{1}{2}}}\,.\,\sinh\left(\frac{ph}{qk}\right)^{\frac{1}{2}}x\Big/\cosh\left(\frac{ph}{qk}\right)^{\frac{1}{2}}x_C \quad . \quad . \quad (1)$$

where v is excess of temperature of rod at the cross section x cm. from the end in contact with the vessel, over that of the vessel, p is the perimeter, q is the area of cross section, h the emissivity, k the thermal conductivity of the rod, and Q is the amount of heat per second crossing the section at any point x_C

If the heat is supplied at a uniform rate per centimetre to the surface of the rod between x_C and the free end, the mean temperature of the surface between x_C and the end may be taken as $v_{C'}$, the temperature which would be observed at a point $x_{C'}$, dividing the distance between x_C and the free end in the ratio 1 : 2, if the equation (1) held throughout the heated

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* *See* Lees, Bakerian Lecture, "Phil. Trans." Roy. Soc. A 208, pp. 381, 443 (1908).

part of the rod. If s is the area of the surface of the rod beyond x_C and H is the total heat supplied to the rod, then

$$Q=H-h\,.\,sv_{C'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

If the temperature excesses v_A and v_B of the rod over the air at two sections x_A and x_B are observed then,

$$k=\left\{\frac{H(x_A S_A-x_B S_B)}{v_A-v_B}-hsx_{C'}S_{C'}\right\}\Big/ q\cosh ax_C \quad . \quad . \qquad (3)$$

where a is written for $(ph/qk)^{\frac{1}{2}}$ and S_A for $\sinh ax_A/ax_A$, &c.

From equation (3) k may be calculated with sufficient accuracy by taking, $\sinh ax/ax=\cosh ax=1$ approximately, and neglecting the term $\left(\frac{hsx_{C'}}{q}\right)$, which, for the experiments described, introduces an error in k of approximately 0·003 per cent.

We thus get

$$k=\frac{H}{q}\,.\,\frac{x_A-x_B}{v_A-v_B}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (4)$$

Since for any one experiment H, q, x_A-x_B, were constant we may take k proportional to

$$(v_A-v_B)^{-1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (5)$$

If R_1 is the resistance of the platinum thermometer used to determine v_A, and r_1 is that for v_B when both are at the jacket temperature, and if R_2 and r_2 are the corresponding values when a temperature difference exists; then it may be shown that

$$v_A-v_B=\frac{1}{B}\left(\frac{R_2}{R_1}-\frac{r_2}{r_1}\right). \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6)$$

where B is a constant.

Hence for any experiment we may take

$$k \text{ proportional to } \quad \frac{1}{q}\left(\frac{R_2}{R_1}-\frac{r_2}{r_1}\right)^{-1}(x_A-x_B)(\text{current})^2 \quad . \quad . \quad . \qquad (7)$$

In order to detect inconsistencies in readings during an experiment a method for finding v_A-v_B approximately was also used.

METHOD OF EXPERIMENTING.

4. When taking a resistance reading for any of the coils, to avoid thermo-effects the galvanometer was kept in circuit and the deflection of the galvanometer noted when the Leclanché cell was connected to the bridge. At the beginning of an experiment tap water was circulated through the jacket and the resistance of the jacket coil was read at intervals. After about

45 minutes the temperatures usually became steady. The resistances of the "hot" and "cold" coils were then read. Then a suitable current was sent through the heating coil. After about 30 minutes the resistances of the coils had become steady and were again read. Owing to slight variations in the jacket temperature it was seldom possible to repeat these readings and obtain exactly the same values. Consequently an attempt was made to read the resistances of the two coils with as small a time difference as possible. A number of such double readings was taken and the mean temperature difference of the coil calculated The resistance of the jacket coil was read at intervals to detect abnormal changes. The wire was then stretched by a force about 0·7 of the elastic limit and the resistances of the coils read again. The stretching force was then removed and readings again taken. The temperature difference between the hot and cold coils varied between 10°C. and 20°C.

The wires tested were all (except the nickel one, which was No. 16 gauge) of No. 14 gauge. When new wires were used the results for successive experiments were not consistent with each other until about the third experiment, owing probably to initial stresses in the wires; and the results were more consistent during the summer months, when the temperature of the jacket water was not affected by the heating apparatus. The effect of the stretching on the dimensions of the wire is neglected, as its influence on the conductivity is too small.

5. Comments on the Results.

For all the wires used, stretching produced a slight increase in thermal conductivity. Although difficulty was experienced in getting results which were consistent throughout, those obtained when conditions were most satisfactory show an increase of about 0·5 per cent. for a tension about 0·7 of the elastic limit. A few experiments apparently showed a decrease in conductivity of the order 0·1 per cent. with stretching, but in these cases the readings were always unsteady and unsatisfactory. In all cases when readings were steady an increase was observed. Several attempts were made to determine the change in conductivity for smaller tensions, and the results seem to indicate that the change is not appreciable until the tension approaches the elastic limit. After the tension was withdrawn the conductivity returned approximately to its original value, although in most cases the value was slightly in excess

OBSERVATIONS AND RESULTS.

Date. 1914.	Duration of experiments. Hrs.	Tensions.		$\left\{\frac{R_2}{R_1}-\frac{r_2}{r_2}\right\}^{-1}$ proportional to k.	Percentage increase in k.	Current used to give H. Amps.	$\left\{\frac{R_2}{R_1}-\frac{r_2}{r_1}\right\}^{-1}$ ×(current)^2 ×(x_A-x_B).
		Kilos.	Kilos/cm.^{2}				
Copper Wire, No. 14 *Gauge.* $x_A-x_B=4{\cdot}25$ cm.							
July 8th...	4	0 15 0	463	26·88 26·98 26·90	0·37	0·3	10·28 10·32 10·29
Sept. 8th .	5	0 10 0	309	16·03 16·12 16·09	0·56	0·4	11·03 11·09 11·07
Sept. 9th .	6	0 10 0	309	15·82 15·84 15·82	0·13	0·4	10·88 10·90 10·88
Sept. 10th.	5	0 10 0	309	27·10 27·29 27·12	0·70	0·3	10·37 10·44 10·38
Sept. 11th.	5	0 10 0	309	27·03 27·25 27·10	0·81	0·3	10·34 10·42 10·37
Steel Wire, No. 14 *Gauge.* $x_A-x_B=4{\cdot}45$ cm.							
Oct. 21st .	4½	0 40 0	1,235	16·66 16·74 16·69	0·48	0·2	2·966 2·980 2·971
Oct. 28th .	4½	0 40 0	1,235	16·57 16·77 16·66	1·21	0·2	2·949 2·985 2·966
Nov. 4th .	4½	0 40 0	1,235	16·36 16·46 16·36	0·61	0·2	2·912 2·929 2·912
1915.	*Nickel Wire, No.* 16 *Gauge.* $x_A-x_B=4{\cdot}75$ cm.						
June 23rd.	4	0 15 0	723	25·46 25·51 25·21	0·20	0·16	3·095 3·101 3·066
June 24th.	4	0 20 0	964	29·33 29·40 29·35	0·24	0·16	3·567 3·575 3·569
June 30th.	3½	0 20 0	964	25·51 25·66 25·50	0·59	0·16	3·104 3·120 3·100

Date. 1914.	Duration of experiments. Hrs.	Tensions. Kilos.	Tensions. Kilos/cm.^{2}	$\left\{\frac{R_2}{R_1}-\frac{r_2}{r_1}\right\}^{-1}$ proportional to k.	Percentage increase in k.	Current used to give H. Amps.	$\left\{\frac{R_2}{R_1}-\frac{r_2}{r_1}\right\}^{-1}$ $\times$(current)^2 $\times(x_A-x_B)$
			Aluminium Wire, No. 14 *Gauge.* $x_A-x_B=4{\cdot}55$ cm.				
Nov. 18th.	4½	0		14·64		0·3	5·995
		15	463	14·68	0·27		6·012
		0		14·68			6·012
Nov. 25th.	4½	0		18·66		0·26	5·740
		15	463	18·75	0·48		5·768
		0		18·66			5·740
Dec. 2nd...	3	0		18·94		0·26	5·826
		15	463	19·18	1·27		5·899
		0		18·79			5·780
			*Brass Wire, No.*14 *Gauge.* $x_A-x_B=4{\cdot}6$ cm.				
Dec. 16th.	4¼	0	463	15·51		0·2	2·854
		15		15·55	0·26		2·861
		0		15·44			2·841
Jan. 27th, 1915	4	0	463	15·20		0·2	2·797
		15		15·25	0·33		2·805
		0		15·19			2·795
1915.			*Zinc Wire, No.* 4 *Gauge.* $x_A-x_B=4{\cdot}65$ cm.				
Mar. 3rd...	4	0		21·82		0·2	4·059
		15	463	21·90	0·37		4·072
		0		21·72			4·040
May 5th...	4½	0		21·17		0·2	3·938
		15	463	21·23	0·28		3·950
		0		21·21			3·945
May 12th.	4	0		21·93		0·2	4·079
		15	463	22·03	0·46		4·098
		0		unsatisfactory			unsatisfactory
May 19th.	3½	0		22·52		0·2	4·189
		15	463	22·68	0·71		4·218
		0		22·62			4·207
May 26th.	4	0		22·93		0·2	4·265
		15	463	22·96	0·10		4·271
		0		22·94			4·267

of the original value. The numbers quoted for k in the results are values proportional to the thermal conductivity.

The relative conductivities calculated from the values of the heating currents and of $x_A - x_B$, on the assumption that the conductivity of copper is 0·90 are: copper 0·90, steel 0·25, nickel 0·18, aluminium 0·50, brass 0·24, zinc 0·35, whose relation is of the right order.

The experiments described in this Paper were carried out at the suggestion of Prof. Lees, to whom I am indebted for the general design of the apparatus used and for valuable suggestions during the course of the work.

ABSTRACT.

In the experiments carried out, the wire to be tested was fixed at each end between the jaws of screw clamps joined to brass cylinders, through which water could be circulated. Attached to one clamp was a rod passing through a framework, enabling tension to be applied by a nut working on a thread.

Heat was supplied to the centre of the wire by passing a current of electricity through a manganin coil wound on the wire, and the temperature difference between two points on the same side of the centre was ascertained by two platinum coils, also wound on the wire.

The resistances of the coils were determined by means of a Callendar-Griffiths bridge, enabling a temperature difference of 0·01°C. to be read, and making it possible to detect a difference in conductivity of 0·05 per cent. If R_1 is the resistance of the platinum thermometer nearer the heating coil, and r_1 is that of the other platinum coil when both are at the jacket temperature; and if R_2 and r_2 are corresponding values when a temperature difference exists, then it may be shown that the conductivity

$$K \text{ is proportional to } \left(\frac{R_2}{R_1} - \frac{r_2}{r_1}\right)^{-1}.$$

For all the wires used (copper, steel, nickel, aluminium, brass, zinc), stretching produced a slight increase in thermal conductivity. The most satisfactory experiments showed an increase of about 0·5 per cent. for a tension of about 0·7 of the elastic limit.

After the tension was withdrawn the conductivity returned approximately to its original value.

DISCUSSION.

Mr. F. E. **Smith** suggested that it would have been of interest if the author had made simultaneous observations of electrical conductivity on the same apparatus. Had any experiments been made on liquids, such as mercury, for example?

Prof. **Boys** asked if the coefficient would be reversed if compression were applied instead of extension. He realised there would be some practical difficulties in making the experiments.

Prof. **Lees**, in reply, said that there would have been electrical measurements included in the Paper but for the author's absence in the Army. He thought compression measurements could be made, though they had not so far been carried out.

XV. *The Effect of Stretching on the Thermal and Electrical Conductivities of Wires.* *By* CHARLES H. LEES, *D.Sc.*, *F.R.S.*

RECEIVED APRIL 26, 1917.

THE measurements which have been made of the effect of stretching on the electrical conductivities of wires for currents flowing along them by Tomlinson, Gray, Smith, Credner and others * have all shown that the fractional decrease of conductivity produced is approximately proportional to the tension, and for a force of a megadyne per square centimetre is of the order 10^{-6}. This holds in all cases with the possible exception of nickel.†

For loads up to the limit of elasticity of the material of the wire the change of electrical conductivity which can be produced does not exceed a few parts in a thousand.

When we turn from the electrical to the thermal conductivity we find that few measurements of the effect of stretching on it have been made. When Mr. Johnstone commenced his work on the subject the only information available was that given by Smith‡ for iron, steel, copper and brass. He found that the effect of loads up to the elastic limit was to increase the thermal conductivities by amounts of the order 2 to 7 per cent. The effect of the same load on the thermal was from 4 to 13 times that on the electrical conductivity.

In general the effect of any change in the purity or physical nature of a material has been found to be greater on the electrical than on the thermal conductivity. Thus, the differences in the electrical conductivities of the samples of nearly pure metals used by Jäger and Diesselhorst and by myself § are considerably greater than those in the thermal conductivities.

For alloys Schulze || and Eucken and Gehlhoff ¶ have shown that the decrease in electrical conductivity as compared with the conductivities of the constituents is much more marked

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* References up to 1900 will be found in Wiedemann's "Elektricität," Bd. 1, and in Winkelmann's "Handbuch der Physik," Bd. 4. Later references are: Williams, "Phil. Mag." XIII., p. 635 (1907); Ercolini, "N. Cimento," XIV., p. 537 (1907); Smith, "Physical Review," XXVIII., p. 107 (1909); Credner, "Zeits. Phys. Chem.," LXXXII., p. 457 (1913).

† Tomlinson's coefficients of 10^{-6} are, for iron, platinum, copper, aluminium, magnesium and platinum-silver about 2; for silver, zinc and platinum-iridium about 4; for tin, 11; for lead, 17; and for nickel, —3; and subsequent work has not modified these numbers seriously. Wiedemann, "Elektricität," 1, p. 501.

‡ Smith *l. c.*

§ Compare the differences shown in Fig. 3 and plate 30 of the Bakerian Lecture, "Phil. Trans.," A. 208, p. 381 (1908).

|| Schulze, "Ver. d. Phys. Ges.," XIII., p. 856 (1911).

¶ Eucken and Gehlhoff, "Ver. d. Phys. Ges.," XIV., p. 169 (1912).

than the decrease in the thermal conductivity, and that in consequence the quotient of the thermal by the electrical conductivity, which is of prime importance in the electronic theories of conductivity, increases as the alloy deviates in composition from either of its pure constituents.*

If we compare the two ranges of values within which the thermal and electrical conductivities of different materials lie, we find that the range for thermal is much less extensive than that for electrical conductivities. Thus, the thermal conductivities of glass and copper or silver are as 1 to 400, while the electrical conductivities are as 1 to 6×10^{19}. Or, confining ourselves to metals for the present purpose, the thermal conductivities of bismuth and copper or silver are as 1 to 50 while their electrical conductivities are as 1 to 70.

General considerations such as these made me hesitate to accept Smith's high results for the effect of stretching on the thermal conductivities of the metals he used, and induced me to get Mr. Johnstone to measure the effect by the method he has described in the previous Paper. His results confirm those of Smith, as may be seen from the following table calculated from the observations now available :—

Fractional Increase of Conductivity Due to Stretching.

Material.	Stress. $\frac{\text{megadyne.}}{\text{sq. cm.}}$	Thermal conductivity.		Electrical conductivity.		Observer.
		Observed.	per $\frac{\text{megadyne.}}{\text{sq. cm.}}$	Observed.	per $\frac{\text{megadyne.}}{\text{sq. cm.}}$	
Copper...	1,580	0·0080	$5\ 10^{-6}$	−0·0025	$-1{\cdot}6\ 10^{-6}$	Smith.
	1,150	0·013	$11\ 10^{-6}$	...	...	Smith.
	303	0·0049	$16\ 10^{-6}$	...	...	Johnstone.
	...	...	...	...	$-2{\cdot}3\ 10^{-6}$	Tomlinson.
Zinc	454	0·0038	$8{\cdot}4\ 10^{-6}$	...	...	Johnstone.
					$-4{\cdot}4\ 10^{-6}$	Tomlinson.
Aluminium	454	0·0067	$15\ 10^{-6}$	...	...	Johnstone.
					$-1{\cdot}9\ 10^{-6}$	Tomlinson.
Nickel ...	946	0·0037	$3{\cdot}9\ 10^{-6}$	...	...	Johnstone.
					$3{\cdot}2\ 10^{-6}$	Tomlinson.
Iron	2,300	0·025	$11\ 10^{-6}$	−0·0031	$-1{\cdot}4\ 10^{-6}$	Smith.
	1,150	0·018	$16\ 10^{-6}$	...	...	Smith.
					$-2{\cdot}1\ 10^{-6}$	Tomlinson.
Steel	5,030	0·026	$5{\cdot}2\ 10^{-6}$	−0·0018	$-0{\cdot}4\ 10^{-6}$	Smith.
	2,300	0·030	$13\ 10^{-6}$	...	...	Smith.
	1,210	0·0077	$6{\cdot}4\ 10^{-6}$	...	...	Johnstone.
Brass ...	2,590	0·022	$8{\cdot}5\ 10^{-6}$	−0·0028	$-1{\cdot}1\ 10^{-6}$	Smith.
	1,170	0·016	$14\ 10^{-6}$	...	...	Smith.
	454	0·0030	$6{\cdot}6\ 10^{-6}$	...	...	Johnstone.

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* It should be noted, however, that Reitzsch, "Ann. der. Phys.," III., p. 403 (1900), found that the addition of small quantities of phosphorus or arsenic to copper reduced its thermal in a slightly greater ratio than its electrical conductivity.

Although the larger of the tensions used by Smith are outside the elastic limits for the materials, there is as fair an agreement as could reasonably be expected between the coefficients of the changes of conductivity per megadyne per square centimetre of stress as calculated from his observations, those of Johnstone for the thermal conductivity and those of Tomlinson for the electrical conductivity.

If we except the change of the electrical conductivity of the hard nickel used by Tomlinson, which was probably far from pure, the results so far available appear to justify the statement that when the commoner metals and alloys are subjected to a tension not exceeding the elastic limit for the material, the thermal conductivity in the direction of the tension is *increased* at a rate between 4 and 10×10^{-6} of itself per megadyne per square centimetre of section, while the electrical conductivity is *decreased* under the same circumstances at about a tenth of the above rate. The quotient of the thermal by the electrical conductivity increases at a rate between 4 and 11×10^{-6} of itself per megadyne of stretching force per square centimetre.

XVI. *Cohesion.* (*Third Paper.*) *By* HERBERT CHATLEY, *D.Sc.*(*Lond.*)

RECEIVED MARCH 15, 1917.

THE objects of this Paper are two-fold:—

1. To restate and add further evidence in favour of an electrical theory of cohesion.
2. To provide certain tentative forms of empirical formulæ for the expression of the inter-molecular forces.

Definitions.

For the purposes of this inquiry, cohesion is defined as follows:—

Cohesion is the net attraction (*i.e.*, balance of attraction over repulsion between molecules which are relatively chemically saturated, at distances not greatly exceeding the molecular diameters. Adhesion is the special case of cohesion where the molecules are dissimilar in constitution.

The Electrical Theory of Cohesion.

In B, p. 312, it is suggested that "Presumably the enormous electrostatic forces which exist round free atoms are not wholly neutralised in the formation of molecules, and cohesion is due to what electricians call "stray field."

On the score of magnitude the cohesive attraction fades into chemical affinity (electrostatic linkage) as the proximity increases, and diminishes into Newtonian gravitation as the particles are separated.

The "stray field" of a bi-atomic molecule may be conceived in various ways according to the manner of linkage, *e.g.*, with or without an electron as the medium. In any case the whole field must be polarised, and the stray field will be expressible in vector terms. Since the electrostatic bond between two monovalent ions in contact is about 10^{-4} dynes, the cohesive attraction should be less than this. Furthermore, at a distance of about two molecular diameters the field must become almost uniform, and at ten diameters simple Newtonian conditions must exist.

The Magnitude of Electrostatic and Newtonian Linkages.

The following table indicates roughly the magnitude of the linkages conceivable between different simple bodies.

The diameter of an electron has been taken as $3{\cdot}0 \times 10^{-13}$cm.

The mass of an electron as $0{\cdot}61 \times 10^{-27}$ grams.

The diameter of an atom as

$$\sqrt[3]{\frac{1{\cdot}6w}{s}} \,.\, 10^{-8} \text{ cm.}$$

where w is the atomic weight and s is the density at absolute zero.

No.	Description of pair.	Position.	Distance c. to c. Cms.	Electro-static force or chemical affinity. Dynes.	Newtonian force. Dynes.	Ratio.
1	Two electrons..............	In contact......	$3{\cdot}0 \times 10^{-13}$	$-1{\cdot}3 \times 10^{6}$	$2{\cdot}7 \times 10^{-37}$	$0{\cdot}5 \times 10^{43}$
2	Do. do.	1 cm. apart ...	1·0	$-1{\cdot}6 \times 10^{-19}$	$2{\cdot}5 \times 10^{-62}$	$0{\cdot}5 \times 10^{43}$
3	One positive atom and one electron	In contact......	$0\ 7 \times 10^{-8}$	$2{\cdot}0 \times 10^{-3}$	$1{\cdot}3 \times 10^{-42}$	$1{\cdot}5 \times 10^{39}$
4	One positive atom (nascent hydrogen) and one negative similar atom .	In contact......	$1{\cdot}5 \times 10^{-8}$	$5{\cdot}1 \times 10^{-4}$	$7{\cdot}6 \times 10^{-40}$	$0{\cdot}7 \times 10^{36}$
5	Do. caesium diatomic ions	In contact......	$6{\cdot}1 \times 10^{-8}$	$1{\cdot}3 \times 10^{-4}$	$8{\cdot}0 \times 10^{-37}$	$0{\cdot}2 \times 10^{33}$
6	Do hydrogen ions ...	Gas at N.T.P. .	$7{\cdot}6 \times 10^{-6}$	$2{\cdot}0 \times 10^{-9}$	$3{\cdot}0 \times 10^{-45}$	$0{\cdot}7 \times 10^{36}$
7	Do. do. ...	1 cm. apart ...	1·0	$1{\cdot}6 \times 10^{-19}$	$2{\cdot}5 \times 10^{-55}$	$0{\cdot}7 \times 10^{36}$

NOTE RE No. 5.—This has been chosen because w/s is a maximum for caesium, and it has been made diatomic so as to indicate the dimensional effect of a molecular interval as distinct from an atomic one.

The mean free path of a nascent atom or a molecule in a gas at N.T.P. is taken as $7{\cdot}6 \times 10^{-6}$ cm.

The Magnitude of the Cohesive Force.

The net value of the cohesive attraction under certain given conditions of expansion is known from tensile tests. Surface tension and the energy of liquefaction also help toward this end. The mean repulsion in the case of a permanent gas also shows what is required to maintain separation.

For iron, in A, p. 447, a molecular linkage of $5{\cdot}0\times10^{-8}$ dynes is suggested. In B, p. 310, a value of $2{\cdot}0\times10^{-7}$ dynes is employed. If a tetratomic molecule is considered and the ultimate stress is taken as 2,000,000 grammes per square centimetre the value, neglecting reinforcement by adjacent molecoles, is $2{\cdot}5\times10^{-6}$. The heat of liquefaction indicates a mean resistance of $1{\cdot}0\times10^{-5}$ dynes per molecular pair, if one half the energy is used in producing potential energy, but, of course, this is very approximate.

If the tensile strength of ice is taken as 10 kilogrammes per square centimetre, and the molecules are $3{\cdot}0\times10^{-8}$ cm. in diameter, then the bond per pair is about 10^{-8} dynes.

Similarly, it will be found that if the energy of a gas molecule is converted to potential during a travel through half the mean free path, the mean resistance is about 10^{-8} dynes.

It appears, then, that the molecular bond at the condition of rupture is from 10^{-8} to 10^{-6} dynes The attraction is therefore slightly more than this, and at the point of equilibrium the repulsion must also be more than this, since it balances the attraction, and both increase with proximity.

Tentative Formula for the Attraction.

Pending a better rule, the following formula is proposed for the attraction:—

$$t_2=\frac{Gm^2}{d^{\left(2+\frac{4d_0}{d}\right)}},$$

where G is the Newtonian constant of gravitation,
m is the mass of a molecule,
d is the molecular interval (centre to centre),
d_0 is the molecular diameter.

At absolute zero, $d=d_0$ and the exponent becomes 6, and $t_2=Gs^2$, where s is the density.

For iron this makes the gross attraction $3{\cdot}9\times10^{-6}$ dynes when the molecules are in contact, $4{\cdot}4\times10^{-9}$ when the extension is 10 per cent., and becomes almost identical with gravity when the extension is 1,000 per cent. The ratio to Newtonian force falls from $1{\cdot}68\times10^{-37}$ to unity as the extension proceeds.

For ice the contact bond is $7{\cdot}2\times10^{-8}$ dynes, falling to $4{\cdot}1\times10^{-13}$ for a 20 per cent. extension. The contact condition gives a ratio to Newtonian force of $1{\cdot}2\times10^{29}$.

This formula thus gives a bond descending from about one-hundredth of the electrostatic force to the ordinary Newtonian values as the proximity decreases from absolute contact to some ten molecular diameters.

The appropriate repulsion, which is presumably largely due to kinetic energy, must decrease with great rapidity as the molecules are separated. It will differ for different molecules, and for iron a form

$$t_1=Kd^{-\gamma_1},$$

where K is about 10^{-500} and γ_1 is about 70, seems to apply.

As collateral evidence should perhaps be included the observed facts that current gradually produces brittleness in metal wires, and that cohesive strength is slightly decreased during conduction. An electrified liquid evaporates more rapidly than a neutral one, and most substances expand slightly when charged. Further than this, it may be suggested that the production of frictional electricity is due to the disturbance or rupture of cohesive fields.

REFERENCES.

A. "The Cohesion of Solids," "Proc. Phys. Soc.," Vol. XXVII., Part V., Aug. 15, 1915.

B. "Cohesion" (Second Paper), Vol. XXVIII., Part V., Aug. 15, 1916.

ABSTRACT.

(*a*) To re-state and add further evidence in favour of an electrical theory of cohesion.

(*b*) To provide tentative empirical formulæ for the expression of inter-molecular forces.

The author defines cohesion as the net attraction (*i.e.*, balance of attraction over repulsion) between molecules which are relatively chemically saturated, at distances not greatly exceeding the molecular diameters, and the following formula is proposed for this attraction :

$$t_2=Gm^2/a^{(2+4d_0/d)},$$

where G is the Newtonian constant of gravitation, m the molecular

mass, d the molecular interval (centre to centre), and d_0 is the molecular diameter.

DISCUSSION.

Dr. R. S. WILLOWS (communicated remarks) pointed out that the statement that an electrified liquid evaporated more quickly than a neutral one is not true in all cases. C. T. R. Wilson's fog experiments show, for small drops, that evaporation is most rapid when the drops do not contain an electric ion.

XVII. *A Note on the Graphs of the Bessel Functions of Integral Order.* *By* B. HAGUE, *B.Sc., D.I.C.*

THE differential equation $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}+(x^2-n^2)y=0$ is of very frequent occurrence in many branches of mathematical physics and it is shown in the standard treatises * that the solution of the equation is expressible in terms of special functions, known as Bessel functions. The solutions of the equation met with in the commonest practical problems are composed of functions of the first kind, defined by the infinite series,

$$J_n(x)=\sum_{r=0}^{r=\infty}\frac{(-1)^r}{\Gamma(n+r+1)\,.\,\Gamma(r+1)}\,.\left(\frac{x}{2}\right)^{n+2r},$$

where Γ is the Eulerian gamma function, and n is generally a positive integer.

In a recent investigation the writer had occasion to plot the graph of the function for certain integral values of n, and the resulting curves seemed so interesting that a complete series of graphs was prepared for all integral values of n from zero to 12 inclusive, using the tables recently published by the British Association Committee.

In the hope that the curves may be of interest to others engaged in practical work the author submits Figs. 1, 2 and 3 for the consideration of the Society. In Fig. 1, $J_n(x)$ for $n=0, 1, 2,$ and 3 is shown; in Fig. 2 the series is extended up to $n=7$; and in Fig. 3 is completed to $n=12$. On the latter diagram a few points are shown of $J_{18}(x)$ in order to emphasise the striking and well-known tendency of $J_n(x)$ towards zero as n becomes large.

Apart from the purpose for which the graphs were constructed, they have also proved a useful means of verifying by graphical means the commoner properties of the functions and some of the simpler solutions of Bessel's equation.

The mathematical theory of the J functions has been investigated with the greatest thoroughness, and tables of the values of the functions have been published from time to time. However, with the exception of J_0 and J_1—the graphs of which are

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* Forsyth, "Differential Equations," and Gray and Mathews, "A Treatise on Bessel Functions."

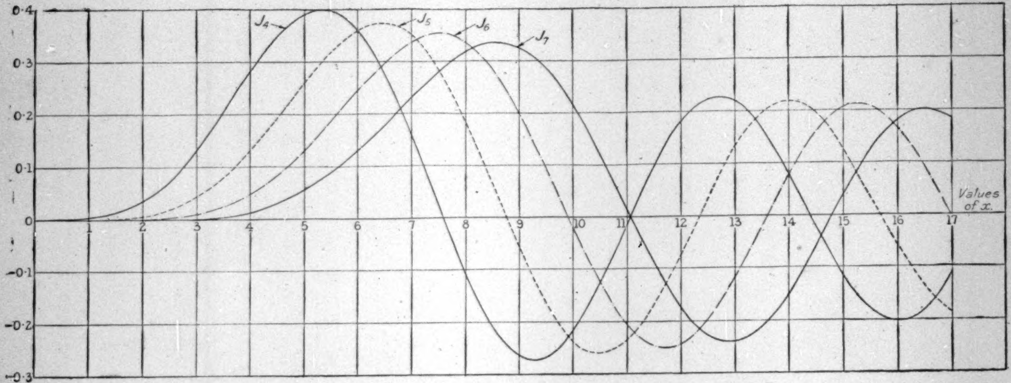

FIG. 2.—BESSEL FUNCTIONS, $J_n(x)$, OF INTEGRAL ORDER. $n=4, 5, 6, 7.$

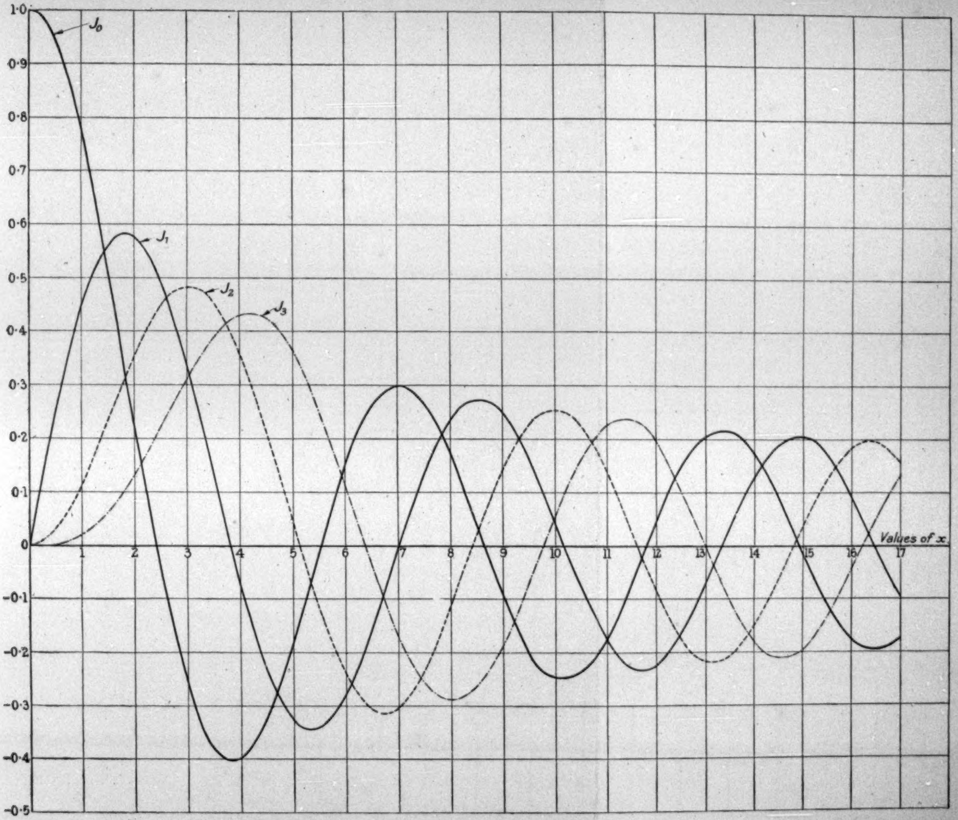

FIG. 1.—BESSEL FUNCTIONS, $J_n(x)$, OF INTEGRAL ORDER. $n=0, 1, 2, 3.$

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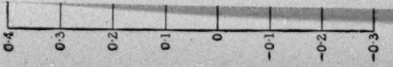

0·4
0·3
0·2
0·1
0
−0·1
−0·2
−0·3

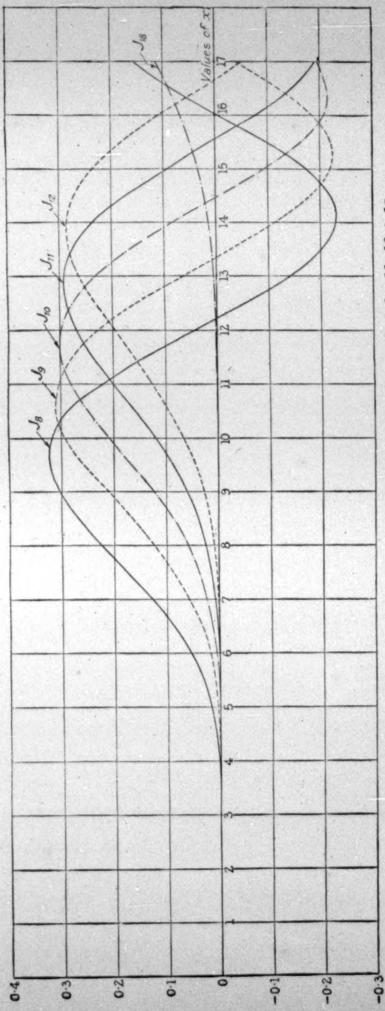

FIG. 3.—BESSEL FUNCTIONS, $J_n(x)$, OF INTEGRAL ORDER. $n = 8, 9, 10, 11, 12$ & 18.

given in Gray and Mathew's treatise and of these and a few other orders in Jahnke and Emdes' "Functionentafeln"—the writer does not recollect seeing the march of the functions illustrated for integral values of the parameter n.

In concluding this short note the author wishes to thank Dr. A. R. Forsyth, F.R.S., for the interest which he has shown in the work, which was carried out at the City and Guilds (Engineering) College.

XVIII. *An Investigation of Radium Luminous Compound.* *By* Clifford C. Paterson, J. W. T. Walsh *and* W. F. Higgins.

Received April 27, 1917.

Synopsis.

1 Introduction.

2. Measurement of Luminosity.

3. Measurement of Radium Content.

4. Measurement of Luminosity of the Markings of Luminous Dials.

5. The Best Dimensions for Luminous Markings to Secure Maximum Legibility.

6. Decay of Luminosity.

    (*a*) Luminous Compounds before Painting.

    (*b*) Painted Dials.

7. Considerations Governing the Choice and Specification of Radium Luminous Compounds.

1. *Introduction.*

In the present Paper is given an account of work carried out at the National Physical Laboratory on Radium Luminous Compounds. The investigations originated in a request by the Admiralty for information and assistance in connection with the use of this material for the dials of large numbers of instruments used by the Navy, and it is by permission of Capt. Creagh-Osborne, Admiralty Superintendent of Compasses, that there are included in this publication details of work which has been carried out at his request. The Compass Department of the Admiralty began the employment of radium luminous compound as far back as 1914, and was the first in the field to adopt it on an extensive scale. The use of separate lighting arrangements for instruments which require to be read at night is thus obviated.

The authors wish to say that there has been no leisure or opportunity to carry out a thorough investigation of the whole subject. Such an investigation would have included inquiry into the origin and characteristics of the various zinc sulphides used and of the effect of impurities on the luminosity of the compound, together with the effect of different methods of infusing into the zinc sulphide the very small quantities of radium which are used. These and similar problems have had to yield place to the necessity for fixing and applying proper standards of luminosity both to the paint as manufactured, and to the instrument dials after the application of the luminous compound.

The standard of luminosity for the paint (generally known as the N P.L. Standard) was fixed in absolute measure in December, 1915, and that for the dials themselves shortly afterwards, and many thousands of luminous instrument dials have since been measured for conformity with the standards laid down.

The subject is a comparatively new one, for the value of radium luminous compound has only been recently recognised in connection with commercial apparatus. Photometric measurements of the brilliancy of zinc sulphide under the action of the radiations from radium products were made by Marsden in 1910,* who directed the radiation from a tube of emanation on to a zinc sulphide screen and measured the luminosity produced. The luminosity was found always to decay at a more rapid rate than the emanation.

In a Paper in 1915,† Mr. H. Hughes, describing the Creagh-Osborne compass, dealt with the use and application of radium compound to prismatic compasses, and in a further Paper to the Optical Society in June, 1916, F. Harrison Glew ‡ dealt with the general question of luminescence, and considered the practical uses to which radium luminous compound may be put, as well as the method of application of the compound to a surface in order to produce the maximum luminosity. Mr. Glew gave a further lecture on this subject to the Illuminating Engineering Society,§ and this was followed by notes on the behaviour and testing of luminous paint by Mr. J. S. Dow and Mr. A. Blok.

A description of the apparatus used at the Bureau of Standards for measuring the luminosity of radium luminous compound has been given by N. E. Dorsey.‖

In the present Paper the ground covered by these authors will only be touched upon in so far as is necessary for the present purpose. It may be recalled, however, that radium luminous compound consists almost exclusively of specially prepared zinc sulphide, with which is mixed about four parts per 10,000 of radium bromide or its equivalent in gamma ray activity.

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* Roy. Soc. "Proc.," LXXXIII., pp. 548-561, 1910. "Sci. Abst.," 76 A., 1911.

† "Improvements in Prismatic Compasses," H. A. Hughes, "Trans." Opt. Soc., Lond., Vol. XVI., p. 17.

‡ "Radium and Other Luminous Compounds," F. Harrison Glew, "Trans." Opt. Soc. Lond., Vol. XVI., p. 276.

§ "Illum. Eng.," p. 72, March, 1917.

‖ "Washington Acad. Sci. J.," 7, pp. 1—6, Jan. 4, 1917.

The luminosity of the resulting powder depends on the amount of radium present and on the quality of the zinc sulphide. The reason for the superiority of one type of zinc sulphide over another is at present obscure, but the responsiveness of different grades of zinc sulphide to radioactive excitation varies very greatly and offers a wide field for investigation. The zinc sulphide responds mainly to the α-ray activity of the radium (*see* Glew, *loc. cit.*). As radium is bought, sold and measured by its γ-ray and not by its α-ray activity, a constant ratio is assumed between the two, and all values of radium content given in this Paper are in terms of γ-ray activity. α rays are emitted from the radium itself and from the emanation, but they are readily stopped by solids (including any organic medium used for binding). According to the so far generally recognised hypothesis the α-particles impinging on the crystals of zinc sulphide excite certain active centres. After its bombardment each active centre becomes extinct and a gradual decay of the zinc sulphide therefore results. That this theory, in its simple form, fails exactly to represent the facts is shown later, and a modification of this theory is discussed in a second theoretical Paper by one of the authors.* To apply the paint in practice the powder is mixed with a minimum possible quantity of varnish or other suitable binding material, and laid on either with a brush or by filling in the figuring which has been engraved into the surface of the object to be painted.

2. *Measurements of Luminosity.*

The measurement of the luminosity of radium paint is rendered difficult by two considerations—(i.) the green hue of the light emitted, (ii.) the very low brightness to be measured. It is desirable at the outset to make determinations of luminosity in absolute rather than relative measure, and comparisons have therefore to be made with the ordinary photometric standards.

(i.) The very green hue of the light given by radium compound necessitates the use of a green filter when photometering it. Marsden examined the light spectroscopically, and found that the visual region lay between $\lambda=5{,}920$ and $\lambda=4\ 250$, with a fairly sharp edge towards the red. It is found that Wratten Filter "Minus Red," used in conjunction with a tungsten filament lamp operating at about 1·5 watts per candle, gives an approximate colour match with the samples of radium paint ordinarily used. A still better match is obtained by the use of a sample of glass made by Messrs. Chance Bros., known as "B.O.T. Light Green," and corresponding with the lightest shade of green glass permitted for ships' starboard navigation lights. The expression of the luminosity in absolute measure entails the determination of the transmission coefficient of the filter. This has been carried out at the Laboratory by a number of observers, so that the result may correspond as nearly as possible with that which would be given by the "average" eye.

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* "The Theory of Decay in Radioactive Luminous Compounds," J. W. T. Walsh.

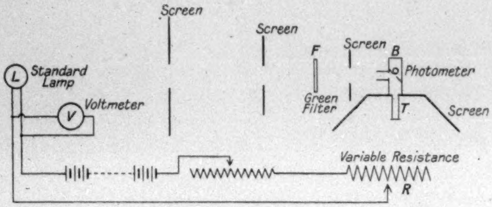

FIG. 1.—DIAGRAM OF APPARATUS FOR MEASURING THE BRIGHTNESS OF RADIUM LUMINOUS COMPOUND.

(ii.) The degree of brightness required to be measured is of the order of 10 microcandles per square centimetre, or 0·03 equivalent foot-candles.*

This is about one-hundredth of the brightness of a newspaper satisfactorily illuminated for comfortable reading by artificial light. The brightness is of the order of that of white paper illuminated by the light of the full moon. It will be seen that this degree of illumination is one at which the Purkinje effect is operative, so that the transmission coefficient of the green filter must be determined at an illumination corresponding with the brightness of the paint with which it is to be used, and it must be ensured that the surfaces to be compared are really in juxtaposition without any intervening space to separate their images on the retina.

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* This is a convenient unit of brightness, and represents a brightness equal to that of a perfectly diffusing surface with a coefficient of reflection of 100 per cent., when illuminated to an intensity of 1 ft.-candle (*i.e.*, by a source of 1 candle placed 1 ft. away from the surface in the direction of its normal).

The following description and diagram (Fig. 1) show a form of apparatus which 18 months use at the Laboratory has shown to be eminently satisfactory for this class of measurement.

L is a tungsten filament lamp mounted on the carriage of a 30 ft. long photometer bench. It is fed from a storage battery, the pressure at the lamp terminals being controlled by a resistance, R, which is to the hand of the observer. The value of the pressure applied to the lamp terminals is indicated by a voltmeter, V. F is the special green filter. It reduces the hue of the light given by L to that of the radium compound. B is a small brass box, 120 mm. long by 37 mm. square, and shown in more detailed plan in Fig. 2.*

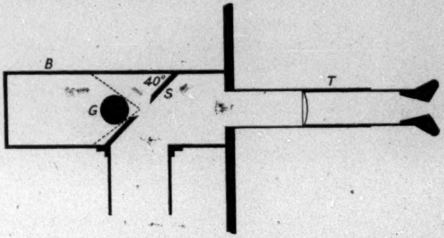

FIG. 2.—DETAIL OF BOX CONTAINING THE COMPARISON SURFACE AND TUBE OF COMPOUND.

It contains S, a sheet of matt white celluloid, set at an angle of 50 deg. to the axis, and having at its centre an oblong aperture, 4·5 mm. high by 7 mm. broad. The relative dimensions are such that when the screen is looked at through the telescope the aperture has the appearance of a small square with 4·5 mm. sides. This aperture is cut with a sharp bevel, so as to reduce to the smallest possible dimensions the black edge between the illuminated screen and the surface of the radium compound.

The radium compound is contained in a glass tube, G, and wedge-shaped guides are arranged inside the box to assist in placing it centrally behind the aperture in S, no matter what may be the diameter of the tube.

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* A very similar apparatus was described by Mr. Blok to the Illuminating Engineering Society, March, 1917.

The transmission coefficient of the filter F was determined from the mean results of five experienced photometric observers, each of whom matched the brightness of three separate samples of radium paint against the brightness of the screen S. The illumination of S was varied by moving the lamp L along the line LF until a match of intensity was obtained. This was done (i) with the filter F removed, and (ii) with the filter in place. The mean ratio of the illuminations was taken to represent, for the conditions under which the apparatus is used, the transmission coefficient of the filter for an "average" eye.

The values obtained for this coefficient by the different observers differed from the mean value by −17·2, −6·4, +4·3, +8·5, and +10·8 per cent. The mean value must not, therefore, be considered as accurate to nearer than 3·5 per cent. This inaccuracy in the absolute transmission coefficient may appear large at first sight, but a very much greater number of observers would be required to reduce it to 1 per cent.

It will be observed that the transmission coefficient was determined under conditions identical with those prevailing when the luminous compound itself is measured—*i.e.*, the size of the field and its illumination are the same—so that those errors are avoided which result from the well-known retinal peculiarities which appear in the use of green light at low illuminations.* The mean coefficient obtained was 2·15 per cent. This value was confirmed by means of a portable photometer, the illumination of a white card being measured directly (*a*) without the filter, (*b*) with the filter interposed.

The other constant to be determined was the coefficient of reflection of the white card S as viewed from the eyepiece T, with the light incident from L at an angle of 40 deg. This was determined by measuring the brightness of the card as viewed from the direction of T with a portable photometer, and comparing this with the brightness of another portion of the same piece of card when held normal to the direction of the light.

The coefficient obtained in this way embodied both the coefficient of reflection K_θ of the card with the light incident at an angle θ, and the reduction of illumination according to the cosine law. This combined coefficient $K_\theta \cos \theta$ was found to be 68·2 per cent.

To obtain now the brightness of a sample of paint *T* from the candle-power C of the lamp L at distance d when the brightnesses of S and T are equal, we have :—

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* "Colour Phenomena in Photometry," J. S. Dow, "Proc." Phys. Soc., Vol. 20, 1906, p. 245.

$$B=\frac{C}{d^2}\times 0{\cdot}21_5\times 0{\cdot}68_2.$$

There is, however, one more point yet to be considered. It will be clear that the transmission coefficient of the filter F will vary with the hue of the light traversing it, so that the value of the coefficient found above holds only for the light from a tungsten filament lamp operating at the one definite efficiency, at which the determination of its transmission coefficient was made. The result of this is that variation of illumination on S has to be obtained by change of position of L along the line LS rather than by change of voltage L. In actual use, however, when measuring the brightness of a sample of radium compound, it is found most convenient to fix L in such a position that an approximate match is obtained in S with a voltage on L of 95. Several settings are then made by means of the resistance R and as the extreme voltage variation on L thus obtained does not generally exceed 2 volts, it is safe to assume that over this range the variation in the hue of the light from L will not appreciably affect the value of the transmission coefficient of F. The candle-power variation with voltage of L has been determined accurately on the photometer bench, over the range from 92 to 98 volts, and from the voltage indicated by V, the distance LS, and the formula given above, the brightness of the paint in T is at once found.

Absorption of Glass Tubes.—The radium paint being in powder form, it is necessary to measure its brightness when enclosed in a glass tube. This is desirable for two reasons. Firstly, it is necessary to obtain a smooth surface of the powder in order that it shall be of uniform brightness, because any cavity in the surface shows a greatly enhanced brilliancy due to internal reflection. Secondly, it is necessary to avoid any possible loss of emanation.

From a series of experiments made when the apparatus was first set up, it was found that variations in the thicknesses of the glass containing tubes, up to at least 1·3 mm. makes no difference to the brightness of the paint viewed through them.

The colouration of the glass by the action of the radium, on the other hand, produces a progressively increasing absorption for which allowance has to be made in considering the decay of luminosity of the compound.* Lead glass gives a brown coloration and soda glass a purple one.

Measurements have been made of the amount of this effect in the case of six of the samples of compound studied, all of which were contained in soda glass tubes. The absorption was found by making alternate measurements of the luminosity of a sample of compound when contained (*a*) in the vessel under examination, (*b*) in a new vessel not previously exposed to γ-radiation. The results obtained are tabulated below and the value of luminosity given throughout the Paper are corrected on the assumption that the light absorption has followed a linear law of increase. As the whole correction does not in any case exceed 16 per cent. this assumption will not lead to any appreciable error in the final corrected values of luminosity.

TABLE I.

Sample.	Age of compound.	Total absorption due to coloration.
3	14½ months.	13 per cent.
6	17 ,,	8 ,,
7	17 ,,	9 ,,
8	11½ ,,	10 ,,
9	13 ,,	10 ,,
10	13 ,,	16 ,,

3. *Measurement of Radium Content.*

The general method of determination of the quantity of radium contained in a concentrated sample is by direct comparison with a standard consisting of a known amount of one of the salts of the element. The means by which this comparison is carried out is based on the ionisation produced by the γ-rays emitted from the samples considered.

Radium gives out three types of rays—namely, the *α*, *β* and γ-rays—and the ionisation produced by any of these could be made use of as a measure of the quantity of radium present. Two difficulties, however, arise in attempting to use either of the two former types of radiation. Firstly, in the case of radium in radioactive equilibrium, it is not possible to isolate these rays, so that the properties of the *α* or *β*-rays alone are employed. The same trouble does not arise in the case of γ-rays as these are very penetrating and may consequently be freed from *α* or *β*-rays by the use of suitable screens of absorbing material. α-rays are absorbed by about 8 cm. of air at atmospheric pressure or by very thin layers of solid matter, while the β-rays are completely stopped by about 1 cm. of lead. In practice it is found that a sheet of lead 5 mm. thick reduces the proportion of β-rays present to a negligible amount compared with the γ-rays. Consequently in the apparatus to be described later the ionisation chamber is covered by a lead disc 5 mm. thick, and the ionisation produced inside this space must therefore be solely due to the γ-rays. In the second place the intensity of all the rays changes with time until complete radioactive equilibrium is attained. Consequently the α or β-ray activity of any sample of radium and its radioactive products will not be sufficiently constant for measurement purposes for some years. For the γ-rays, however, a constant intensity is reached in a comparatively short time (about 30 days suffices) as the products of disintegration beyond radium do not emit γ-rays (with the exception of some very soft rays, which in practice are completely absorbed before the ionisation chamber is reached). For purposes of measurement it is necessary to prevent the escape of the gaseous product—viz., radium emanation—and to this end the material must be sealed up in glass or metal tubes for a sufficient length of time before comparisons can be undertaken.

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* Glew, *loc. cit.*, p. 287.

The γ-ray method is primarily intended for the estimation of high concentration, but may also be extended to estimation of the total quantity of radium in a sample of radium luminous compound if the necessary precautions as detailed later are taken.

The smallest equivalent quantity of radium bromide which can in this way be accurately measured is about 1 mgm., and the quantity of radium luminous compound on which the test is made should hence be from 3 to 5 grms.

The standard against which the samples of radium luminous compound are compared at the National Physical Laboratory, consists of a preparation of pure radium bromide, equivalent to 5·20 mgm. of hydrated radium bromide ($RaBr_2 2H_2O$). This is compared at intervals with the British Radium Standard, which was certified by the International Radium Standards Committee after comparison with the International Standard at Sèvres.

The apparatus employed is represented in Fig 3. It consists of a massive wooden bench along which a stand, *A*, can be moved. The position of this stand with respect to the ionisation chamber is indicated by a pointer and scale. The samples under test are placed in turn in the V-support on the stand *A*, and this is so arranged as to bring the sample approximately to the level of the centre of the ionisation chamber.

The ionisation chamber *C* consists of a lead cylinder of internal diameter 140 mm. and depth 94 mm. The thickness of the walls is 6 mm. The front of this chamber consists of a lead plate of 5 mm. thickness; this is held in close contact with the walls of the chamber by suitable clamping screws. Alternative lead plates of different thicknesses are available as desired. The central insulated disc is supported from the back of the chamber by means of a brass wire passing through an amberite* bush.

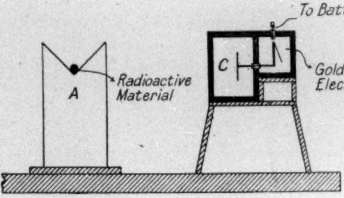

FIG. 3.—DIAGRAM OF APPARATUS FOR MEASURING RADIUM CONTENT BY THE γ-RAY METHOD.

The gold leaf electroscope by which the ionisation current is measured is attached to the ionisation chamber. The gold leaf is fastened on to a vertical projection from the brass supporting wire of the disc above referred to, and is surrounded by a square built up box of thick lead plates. Windows are provided at each side of this box by which the gold leaf may be illuminated and viewed. These windows are of lead glass, and are screened by lead shields to prevent as far as possible the entrance of stray and scattered γ-radiation. The insulated system, consisting of the disc in the ionisation chamber, the gold leaf and the support are positively charged by means of a charging key carried on the top of the electroscope. On pressing down the plunger of this, contact is made with a battery of accumulators of the required voltage (about 200 volts). The case and all metal parts of the apparatus other than the central insulated system are earthed, as is also the negative pole of the battery.

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* This material is composed of highly compressed amber chips or dust, and provides an excellent insulator, which may be turned to any desired shape. Care must be exercised in doing this as the material is somewhat brittle. It is not hygroscopic, nor does it show any appreciable soakage effect.

The motion of the gold leaf is viewed through a reading microscope magnifying about 10 diameters, and provided with a scale in the eyepiece. This instrument is held rigidly by means of an adjustable stand, and when once put into position is not moved throughout the complete series of readings.

The readings are always taken over a definite portion of the eyepiece scale, the time being determined by means of a stop-watch for the gold leaf to move over this distance. The procedure generally adopted in making a comparison is as follows: The natural leak of the system is first determined. This is done by charging up the gold leaf and noting the time taken by it in moving over the measured distance, all radium being removed from the neighbourhood of the measuring instruments. The radium standard is then placed on the stand *A*, which is moved to some definite position along the test bench according to the strength of the sample to be measured. For small quantities the stand is brought relatively near the ionisation chamber, while for larger amounts the distance between the two is increased. The rate of leak is then determined for the standard sample, the mean of a number of observations being taken. The standard is then replaced by the material under test without the relative positions of any parts of the apparatus being changed and the rate of leak is again determined. Further readings of the standard are then taken, which should agree with those first obtained.

The comparison is then repeated for other distances between the stand *A* and the ionisation chamber *C*. Finally, the natural leak is taken at the end of the series of observations. It must be noted that all radium preparations except that actually on the test bench, must be removed as far as convenient from the γ-ray apparatus, otherwise the natural leak will be large, and the corresponding accuracy of the comparisons will be greatly reduced.

The correction for the natural leak is made thus: If T_0 is the time taken for the leaf to move over n divisions when the sample of radium is under observation and T_n the number when it is absent (*i.e.*, the natural leak) then the corrected value of the time (T) will be given by—

$$\frac{n}{T} = \frac{n}{T_0} - \frac{n}{T_n},$$

i.e.,

$$T = \frac{T_0 T_n}{T_n - T_0}.$$

The corrected value of the time is thus obtained for the sample under test and for the standard. The ratio of these times then gives the ratio of the quantities of radium present in the standard and the sample under test, thus :—

$$\frac{\text{Weight of radium in sample}}{\text{Weight of radium in standard}} = \frac{\text{Time for standard.}}{\text{Time for sample.}}$$

Correction for Absorption.—From the above description it will be clear that in determining the radium content of a sample of luminous compound by measurement of its γ-ray activity, allowance must be made for the absorption of the γ-rays by the zinc sulphide. If, first, we suppose that the whole of the radium salt is concentrated along the axis of a cylindrical vessel containing the compound, it is known that the intensity of the γ-ray activity in any direction perpendicular to the axis (and at a distance from it sufficient to make the length of the vessel negligible in comparison with this distance) will be given by $I' = Ie^{-\lambda a}$, where I is the intensity which would be found in the absence of the zinc sulphide, a is the radius of the vessel, and λ is the absorption coefficient of the zinc sulphide.

In actual practice, however, the radium salt is not concentrated at the axis, but is uniformly distributed over the cross-section of the cylinder, so that the above formula does not accurately apply, although *a priori* reasoning from the known value of λ shows that for vessels of small diameter this formula is sufficiently accurate for all practical purposes. When the diameter reaches 20 mm. or 30 mm., however, it becomes necessary to investigate the effect of uniform distribution more thoroughly, and this may be done in one of two ways—(a) approximately, by first treating the simple case of a vessel of square section and then regarding the cylindrical vessel under consideration as made up of any convenient number of equal square-section vessels, laid alongside one another; or (b)

rigidly by obtaining a convergent series with constant coefficients which will give at once the required value of intensity, from a knowledge of the value of λ. Methods (*a*) and (*b*) are described in greater detail in Appendix A

The value of λ has been determined experimentally by mixing a measured quantity of radium salt with an amount of zinc sulphide sufficient to produce radium luminous compound of standard specification, and then determining the γ-ray activity of the resulting mixture. The contents of a tube of radium salt equivalent to 11·4 mgm. of hydrated radium bromide, as determined by a γ-ray test were mixed, without loss, with 28·5 grms. of active zinc sulphide, and the mixture was enclosed in a glass bottle of known dimensions. The γ-ray activity was measured at intervals until a constant value was obtained. This was equivalent to 8·2 mgm. of hydrated radium bromide, and from this result a value for λ is calculated from the formula on page 226. This value is $\lambda=0{\cdot}03\ \text{mm}^{-1}$.

A confirmatory test was made by the use of a second sample consisting of 30 grams of radium luminous compound, which was measured in bulk, and then was subdivided into a series of six smaller quantities enclosed in glass tubes of various diameters.

The γ-ray activity of each of these was determined and was corrected for absorption in accordance with the value obtained in the first test. The radium content per grm. was found to be the same, to within an accuracy of 2 per cent., for each tube and for the sample in bulk when this had been done.

An attempt was also made to obtain the absorption coefficient by surrounding a sealed tube of radium bromide with a known thickness of zinc sulphide as mentioned above. The value so obtained leads to a result of $\lambda=0{\cdot}0092_5\ \text{mm}^{-1}$, which is roughly one-third of the value obtained in the case of uniform distribution. No explanation of this discrepancy is at present available, but further work is proposed to investigate the matter. In the meantime the first value is assumed to hold, as the conditions under which it is obtained most nearly represent the conditions of actual test of samples, and the table following gives the factors by which the measured radium content must be multiplied in order to give the true content for different sizes of containing vessels.

TABLE II.
($\lambda = 0{\cdot}03\ mm^{-1}$).

Diameter of containing vessel.	$\frac{\text{True content}}{\text{Measured content}}$.
5 mm.	1·06
10 mm.	1·13
15 mm.	1·20
20 mm.	1·28
25 mm.	1·35
30 mm.	1·43

4. *Measurement of Luminosity of the Markings of Luminous Dials.*

The chief commercial use of radium luminous compound lies in its application to instrument dials for rendering the figures and graduations on these visible at night without external illumination. The method by which this application may be made has been described by F. H. Glew (*loc. cit.*), and it was early found that the various methods employed by different makers of the dials gave very widely differing results. The luminosity of the figures, therefore, depends not only on the quality of the radium luminous compound employed, but also on the method of its application to the dial, so that the employment of a good quality of luminous compound is not a necessary guarantee of a satisfactory dial.

It becomes necessary then, to test the actual brilliancy of the luminous markings, and the only satisfactory method of doing this is to compare the dial under test with another dial, the markings of which are of the same pattern and dimensions, and of a known brightness. Hence the first test employed was comparison with a radium painted dial of what was considered to be good luminosity. This was very soon abandoned, however, in favour of a more convenient and fundamental test. In the first place the possibility of a deterioration of luminosity was early recognised as one of the dangers of a radium-painted standard dial, while the large number of different patterns of dial to be tested would have necessitated a considerable collection of radium painted standards. These would necessarily have been at variance as regards brilliancy, since absolute uniformity is practically impossible of attainment, especially under the conditions prevailing at the time when the standards were first set up. A still more fatal drawback was the fact that beyond the statement that the test dial did or did not equal in brilliancy the standard dial, no value could be assigned to the luminosity of its markings to serve as a guide for future work by the manufacturer.

In view of all these considerations a method was devised which has from the very first given complete satisfaction in working. This method is, in essence, the comparison of the test dial with a metal stencil cut to the same pattern and dimensions as the luminous markings of the dial, and having behind it a screen illuminated by light of the same hue as that given by the radium compound. The intensity of illumination of this screen is variable by means of a resistance in series with an electric lamp, and thus a knowledge of the variation of illumination of the screen with the current taken by the lamp gives at once the value of the brilliancy of the markings on the test dial. The actual apparatus is shown diagrammatically in Fig. 4, where a box encloses at one end an electric lamp, *L*, in circuit with a resistance, *R*, and precision ammeter *A*. *R* is to the hand of the observer. *L* is actually a 105-volt 60-watt metallic filament lamp operated, when giving standard brilliancy, at about 5 watts per candle. In front of *L* is a sheet of Wratten "Minus Red" filter *F*, in order that the hue of the light may match that of the radium compound The green light then falls on two screens *S*, *S*, each consisting of two thicknesses of thin white paper with a space of 10 mm. between them. The two thicknesses of paper were found necessary to secure proper diffusion of the light and it was also found desirable to whiten the inside of the wooden strips separating the two layers of paper in order to avoid any decrease of luminosity at the edges. In front of the screens are placed two stencils *T*,*T*, each consisting of a sheet of thin brass or copper 0·3 mm. thick, framed in wood to fit the dimensions of

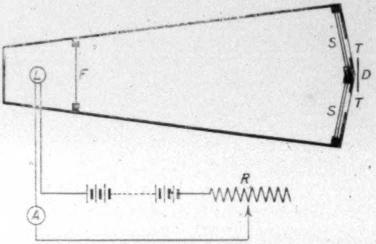

FIG. 4.—DIAGRAM OF APPARATUS FOR MEASURING THE BRIGHTNESS OF THE MARKINGS OF LUMINOUS DIALS.

the screens (255 × 275 mm.). The figures and graduations are cut in brass on an engraving machine, so as to be identical with the luminous markings of the dials under examination. Between the stencils is fixed the dial under test *D*. It was originally intended that the test dial should be placed above the stencils so that the three dials under comparison should be one at each apex of an equilateral triangle. This arrangement was abandoned, however, as it was found during the course of the initial experiments that different luminosities were assigned to a dial according as it was placed above, below, to the left or to the right of the dial with which it was being compared. This effect is apparently one of personal error depending on the distribution of sensitivity to green light over the retina of the observer. In different observers the effect was found to vary in amount, and also to change sign. In these circumstances it was found best to give the dial an intermediate position between two comparison dials of equal luminosity. In making a test of a luminous painted dial by this method the dial, as has been said, is placed between the two stencils of similar pattern, while the operator, seated with his eye on a level with and about 3 ft. from the dial varies the brightness of the comparison stencils by means of the resistance *R*. As soon as he is satisfied that the three dials are of equal luminosity he notes the current indicated by the ammeter *A*, and from a table finds the percentage of standard luminosity of the dial under test.

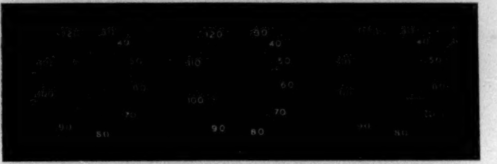

Fig. 5.—Photograph showing the Appearance of a Luminous Dial and its Comparison Stencils in the Dial-testing Apparatus.

This test is, of course, performed in a dark room, and it is necessary for the observer to remain sufficiently long in the dark for his eyes to become "dark-adapted." The appearance of the test dial and its corresponding stencils is shown in the photograph (Fig. 5). It may be here noted that the luminosity of the markings on standard dials is 0·0075 equivalent foot-candles (2·7 microcandles per square centimetre), and that to obtain the above photograph with an Imperial Standard rapid plate an exposure of 2½ hours was required.

An examination of the negative shows that, although there was a luminosity match between the three dials, the light from the centre one, painted with luminous compound, was more actinic.

5. *Investigation of the Best Dimensions for Luminous Markings to Secure Maximum Legibility.*

In addition to the question of luminosity of the markings of a radium painted dial, there is also the consideration of legibility, which will depend on the form and size of the markings as well as on their luminosity. An investigation was undertaken at the request of the Admiralty to determine the influence of these factors.

It is obvious that if the luminous lines or markings of a card are reduced in breadth too much, the letters will become illegible, because the total amount of light has become so small.

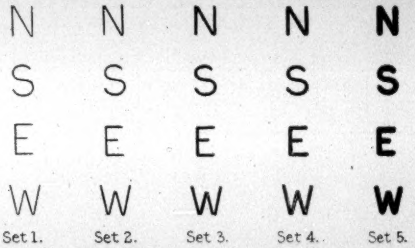

Fig. 6.—The Types of Figures Used to Investigate the Relation Between Dimensions and Legibility.

Conversely, if the lines are increased in width beyond a certain point, in spite of the large amount of light emitted, the letters will lose legibility and appear merely as blotches of light. Between these two extremes there must lie a region where the legibility is a maximum. To determine this optimum width of line, observations were made on a series of letter stencils in which the letters were all 16 mm. high, but in which the width of line varied from 0·5 mm. to 3·0 mm. The letters are shown in Fig. 6. The first set with the thinnest line was used as a standard and the other sets were compared with it. The specific luminosity of each set of letters was independently variable, and was so adjusted that the *total* light emitted was the same for each set. This adjustment was made by reducing the brightness of the broader letter by an appropriate number of thicknesses of fine black tulle. Thus, any two sets of letters when compared from a distance appeared equally bright, and it is of interest to note that under these conditions the difference in the width of the lines could not be detected.

If, however, instead of comparing the letter from the point of view of brightness, the observer made a judgment of the relative legibility, a difference was in some cases noticeable. The set having the broadest lines (set 5) compared with the thinnest (set 1) was clearly less legible. The falling off in legibility was not, however, so noticeable between sets 1 and 4, and was inappreciable between sets 1 and 3.

In order to get a quantitative measurement of the falling off of legibility of sets 5 and 4, their *luminosity* was increased until the *legibility* equalled that of set 1. Equality resulted when set 5 was increased about 40 per cent. and set 4 about 20 per cent. The results are summarised as follows :—

TABLE III.

—	Width of line.	Ratio of width of line to height of letter.	Legibility.
Set 1	0·5 mm.	1/32	All equally legible for equal total luminosity.
Set 2	1·0 mm.	1/16	
Set 3	1·5 mm.	1/11	
Set 4	2·0 mm.	1/8	Fall of legibility = 20 p.c.
Set 5	3·0 mm.	1/6	Fall of legibility = 40 p.c.

It will thus be seen that for a certain specific luminosity, set 5 will not appear very much more legible than set 3, because although set 5 is twice as bright, it suffers a falling off in legibility of about 40 per cent. An experimental trial was made to test these deductions and was found to confirm the results stated.

It was concluded from these experiments that the proportion of width of line to height of letter for maximum legibility should be about 1 to 8 (set 4), and should not much exceed 1 to 7, at any rate for letters and figures in which the height exceeds about 5 mm. Below this limit experience suggests that the breadth of line may be usefully increased to one-sixth of the height of the figure.

6. *Decay of Luminosity.*

Glew points out * that the luminosity of radium compounds falls off with time. The radium itself decays to half its value in about 2,000 years, and it has sometimes been assumed that radium paint will retain its luminosity for a similar period. This is not, of course, the case, as Marsden showed in his experiments on zinc sulphide, willemite and barium platinocyanide. His source of radiation, however, being radium emanation, was itself decaying at a comparatively rapid rate, while the intensity of the radiation far exceeded anything practicable in the preparation of ordinary commercial radium compounds.

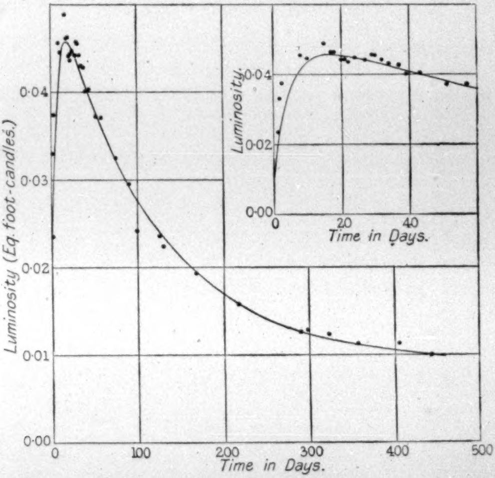

FIG. 7.—DIAGRAM SHOWING THE VARIATION THROUGHOUT LIFE OF THE LUMINOSITY OF SAMPLE 3 OF RADIUM LUMINOUS COMPOUND. INSET, THE EARLY PORTION OF THE LIFE ON AN ENLARGED SCALE.

The authors have made systematic tests to determine as accurately as possible the loss of brilliancy of the radium compound ordinarily used, both in powder form, before painting, and also after mixture with the varnish used in its application to the surface to be treated.

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* *Loc. cit.*, p. 216

(*a*) *Luminous Compounds Before Painting.*—Nine samples from several makers, and containing different proportions of radium salt mixed with various qualities of zinc sulphide have been retained under observation at the National Physical Laboratory. In some cases it has been possible to obtain measurements of the brightness at intervals over a period of about 15 months, and curves showing the rise and fall with time of the luminosity of two characteristic samples are shown in Figs. 7 and 8.

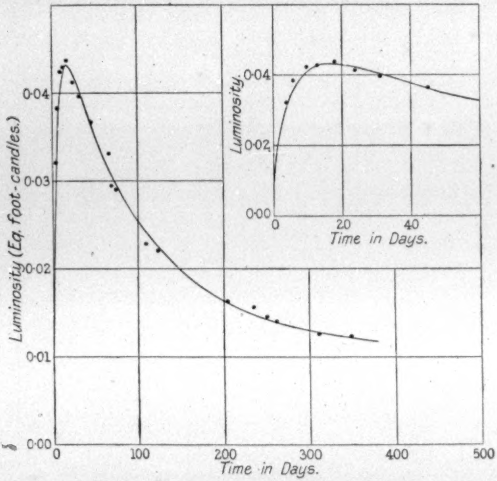

FIG. 8.—DIAGRAM SHOWING THE VARIATION THROUGHOUT LIFE OF THE LUMINOSITY OF SAMPLE 8 OF RADIUM LUMINOUS COMPOUND. INSET, THE EARLY PORTION OF THE LIFE ON AN ENLARGED SCALE.

In samples 3 and 8 (Figs. 7 and 8) measurements were possible within a short time after mixing the radium salt with the zinc sulphide. The curves referred to above have an inset showing in greater detail the earlier history of these samples. It is seen that there is a growth of luminosity to a peak value which is attained generally about 15 days after mixing. The characteristics of the early life of luminous compound are of special interest as bearing on the theory underlying the action of radium on zinc sulphide.

The values of luminosity observed during the first 40 days are taken from the curves and scheduled in the following table :—

TABLE IV.

Period after mixing. Days.	Luminosity (equivalent foot-candles).			
	Sample 3.		Sample 8.	
	Actual.	P.c. of maximum.	Actual.	P.c. of maximum.
4	0·032	71	0·030	70
6	0·038	84	0·035	81
8	0·041	91	0·039	91
10	0·043	95	0·041	95
15	0·045	100	0·043	100
20	0·045	100	0·043	100
25	0·044	98	0·042	98
30	0·043	96	0·040	93
35	0·042	93	0·039	91
40	0·041	91	0·037	86

It should be noted that these samples were from different makers, the method of mixing the radium salt with the zinc sulphide being stated to be different in the two cases. In the first sample the mixture was made with both salts in the dry state, while in the second the radium salt was added in the form of a solution to the zinc sulphide, the resulting mixture being then evaporated to dryness. With this latter method of mixing it is certain that nearly all the radium emanation is driven off, so that the subsequent increase of luminosity of the radium paint is due to the gradual growth of the radium emanation and its products radium A and radium C.

After attaining its maximum value at the end of about 10 to 20 days from mixing, the luminosity then begins to decrease, at first very gradually and then more rapidly until, after a period of about six to seven weeks from the date of mixing, the rate of decay appears to have settled down in accordance with an exponential law—*i.e.*, the rate of decay (k) of luminosity (B) at any time is approximately proportional to the value of the luminosity at that time and $B/B_0=e^{-kt}$. The logarithm of the luminosity therefore plotted against time should give a straight line. The logarithmic curve corresponding with the results of sample 3 (Fig. 7) is shown in Fig. 9. After an initial period of 40 days it will be seen that the logarithmic relation given above is nearly fulfilled up to the end of about 150 to 200 days, but that after this period the luminosity does not diminish so rapidly as would be expected from the above law. It will be shown in a further Paper * that the effect may be explained on the assumption that superposed on the normal decay proportional to the luminosity, there is a smaller effect operating in the reverse direction, which becomes relatively more and more important as the luminosity becomes smaller. The observations have not continued long enough to enable the relative importance of these effects to be determined with great precision, but the results point to the conclusion that after

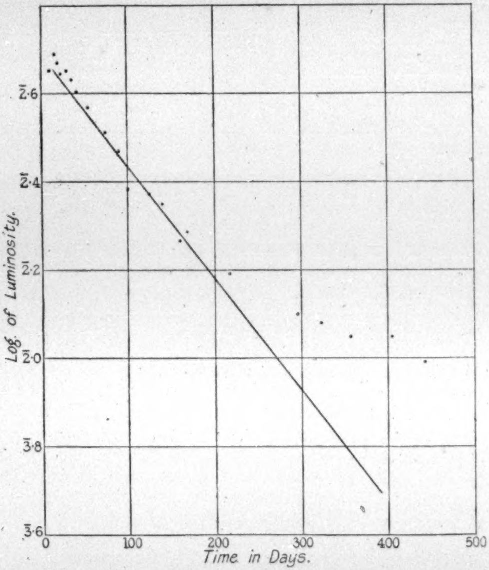

Fig. 9.—Diagram showing the Gradual Departure of the Observed Values of Luminosity of Sample 3 from the Logarithmic Relationship.

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* "The Theory of Decay in Radioactive Luminous Compounds," by J. W. T. Walsh.

about 500 days the luminosity will attain a value which is sensibly constant. This period, however, is not the same for all compounds, as in some of them the rate of decay appears to diminish more rapidly than in others. The effect on the rate of decay of the amount of radium present is shown by the curves in Figs. 10 and 11. Each of the two pairs of compounds 1 and 2 (Fig. 10) and 4 and 5 (Fig. 11) was made up at the same time by the same manufacturer with an identical quality of zinc sulphide. One sample of each pair was prepared with exactly half the quantity of radium salt contained in the other. In the case of the first pair, observations were not made until about 100 days after mixing, while in the case of the second pair the first five months' observations are all that have so far

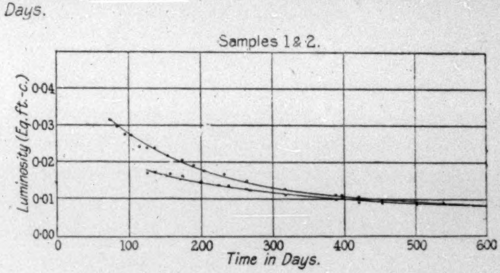


FIG. 10.—LUMINOSITY CURVES FOR SAMPLES 1 AND 2 OF DIFFERENT RADIUM CONTENTS, SHOWING THE SLOWER RATE OF DECAY OF COMPOUND CONTAINING HALF QUANTITY OF RADIUM.

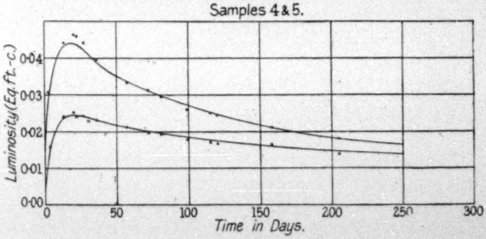


FIG. 11.—THE EARLY PORTIONS OF THE LUMINOSITY CURVES FOR SAMPLES 4 AND 5 OF COMPOUND CONTAINING RADIUM IN THE PROPORTION OF 2 : 1.

been obtained. It will be seen that the initial brightness is in the ratio of the radium content, but the rate of decay of the compound of smaller content is slower, so that the ratio of the luminosities gradually approaches unity as time goes on. This question is one of considerable commercial importance, for it is clear that if the compound is required for use after the lapse of several years, there is very little advantage, except from the point of view of initial brightness, in increasing the radium content beyond a certain point. From present

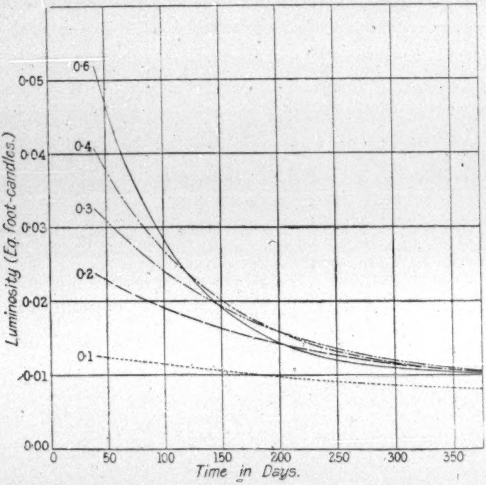


Fig. 12.—Theoretical Luminosity Curves for Samples of Radium Content (mg./gm.) shown against Each Curve. (Constants of Decay assumed of Fair Average Values.)

knowledge it would appear that for short period use (say three to four months) the radium content may be usefully as high as 0·4 mg. per grm. Fig. 12 shows a family of luminosity curves based on the average data at present obtained for compounds made up with different proportions of radium salt in the same quality of zinc sulphide, this sulphide being assumed to have characteristic constants of a fair average value.

It is to be noted that in this diagram in which the effect of the diminution of decay with time is allowed for, the curves for 0·4 and 0·2 mg. compound do not cross one another as would curves based on a purely logarithmic law with constant rate of decay throughout.

(*b*) *Painted Dials.*—Seven dials painted with different samples of 0·4 mg. luminous compound have been kept for the past 12 months, and their luminosities have been found to decay to approximately half value in that period. The actual results of the measurements are given in Table IV. The dials are all ordinary metal dials used for aeroplane instruments, painted with a matt black surface, and having the luminous compound laid on in fine lines about 0·2 mm. to 0·3 mm. high to mark the graduations and figures of the dial. These dials were tested in the apparatus described on p. 229, with the following results :—

TABLE V.

Dial No.	Initial luminosity (p.c. of standard).	Luminosity after 11 months (p.c. of standard).	Decay of luminosity (p.c. of initial luminosity).
1,406	94	60	63
995	82	46	51
1,470	94	58	62
1,783	63	34	54
1,794	24	(16)	(67)
1,496	91	40	44
1,497	52	25	48
		Average	55 p.c.

It is to be remarked that the rate of decay of luminosity of the dials is noticeably lower than that of the compound before application, and as has been said, the initial luminosity of the painted dial is also smaller, being of the order of one-third to one-fourth of that of the compound before application. The reason for this is not clear, but it seems reasonable to suppose that the effect of the varnish or other medium is to impede the activity of the α-ray bombardment from the radium and in this way to cause an effective lowering of the radium content of the compound with a subsequent reduction of both the initial luminosity and also the rate of decay. It is not suggested that the whole effect is due to such a cause, but it seems likely that some such effect as this is operative to account for at least a portion of the observed effect. It is significant, for instance, that the luminosity standards for the dry compound and for the painted dials are 0·03 and 0·0075 ft.-candles respectively (*i.e.*, in the ratio of 4 : 1), while, if we assume curve 0·4 of Fig. 12 to represent the behaviour of the compound before application to dials, the relative behaviour *after* application would be very well represented, both as regards initial brightness and rate of decay, by a curve lying between those for 0·2 and 0·1 of the same diagram.

7. *Considerations Governing the Choice and Specification of Radium Luminous Compound.*

(*a*) *Radium Content.*—It will be clear from what has been stated above concerning the relation between the radium content of a compound and the rate of decay of its luminosity, that, in specifying a standard for any particular purpose, the probable life of the compound in actual use must be the principal deciding factor as to the particular radium content to be employed. The factor of cost, however, must not be ignored. Compound containing 0·4 mg. of radium per gramme of zinc sulphide costs about £6 to £7 per gramme. This is mainly due to the amount of radium present, and the cost may, therefore, be considered as roughly proportional to radium content.

Further, from what has been said above (p. 239) as to the relative rates of decay of radium compound in powder form and after application to dials, it will be seen that in considering the standards to be adopted in the two cases, the radium content of compound *after application* must be considered as reduced to about one-third, in order to ascertain its rate of decay from the data given in the diagram of Fig. 12, p. 238, which refer to the powder only ; that is to say, a *dial* painted with compound containing 0·3 mg. per grm. of radium will follow approximately the same decay curve as *powder* containing 0·1 mg. per grm.

For the compound alone (as distinct from painted dials) after the first six months, the luminosities of compounds of all contents between 0·2 mg. per gramme and 0·8 mg. per gramme are the same to within 20 per cent.

Considering, first, the upper limits desirable for radium content, it is seen that if the compound is likely to be required for more than six months, the use of a compound of radium content greater than 0·2 mg. per gramme can only be justified by a need for greater brilliancy during the early part of the life. Similarly, it may be said that for painted dials the maximum useful radium content for a life of over six months will not be more than 0·6 mg. per gramme.

In the case of instruments of which the life under normal working conditions is appreciably less than six months the use of a compound containing, say, 0·6 mg. per gramme or over might be of advantage, provided it could be ensured (*a*) that the dials were painted within three weeks of the preparation of the compound, and (*b*) that the instruments were not kept in store for more than a month after painting. Probably, in ordinary circumstances, these conditions would not be easy of realisation, and the use, therefore, of a compound of greater radium content than about 0·4 mg. per gramme may be ruled out. Even for 0·4 mg. compound it is important that the interval between the preparation of the compound and its application to the dial should be as short as possible, for it is clear that deterioration in luminosity takes place from twice to three times as rapidly in the powder as in the painted dial. Thus, a deterioration due to one month's delay in painting will be as much as that resulting from about three months' storage of the completed dial.

Referring next to the lower limits desirable for radium content, we see that for cases in which the compound is employed in powder form, the minimum useful radium content is only a little below 0·2 mg. per gramme, and except on the score of expense alone it is not desirable to employ a compound of less radium content than this, because the luminosity of the 0·1 mg. compound after an indefinitely long period, is still nearly 20 per cent. below that of the 0·2 compound. Hence, even in cases where very long life is desirable without the possibility of replacement, the use of the higher grade material is to be preferred unless the cost of the radium compound used has a preponderating influence on the price of the complete instrument. In such cases, rather less than half the cost would be saved and 20 per cent. of the final luminosity would be sacrificed by the adoption of 0·1 mg. per gramme for the compound in powder form.

Similarly for compound used in painting dials, 0·4 mg. per gramme is the best content to use if final luminosity is not to be sacrificed. Some 50 per cent. saving of cost should result from the use of compound for painting dials containing 0·2 mg. per gramme, but the final luminosity would be somewhat less. It should be remembered that there is nothing appreciable to be gained by the use of compounds for dials *below* this last named limit because it will be nearly as economical, and a better result will be achieved by employing a smaller quantity of the better grades of compound instead of the large quantity of a compound of lower radium content.

(*b*) *Luminosity.*—The standard of luminosity for the luminous compound in powder form should be so fixed as to enable a good grade of compound to pass the test within a reasonable period after its preparation. The present standard for 0·4 mg. compound (*viz.*, $0{\cdot}03_0$ f.c.) is easily attainable in a good grade three months after preparation. This time limit may be extended for grades of lower content to, say, four months for the 0·2 mg. compound and six months for 0·1 compound. The corresponding luminosity standards for the three grades 0·4, 0·2 and 0·1 are then 100, 60 and 35 per cent. of the present National Physical Laboratory standard. For painted dials the standards of luminosity should be 25 per cent. of the standards fixed for the powders from which they are to be painted. The above-named figures of 100, 60 and 35 per cent. will then express the standards of the painted dials in terms of the present National Physical Laboratory standard for dials—*i.e.*, 0·0075 equivalent ft.-candles.

The above conclusions have been based on the results obtained by observation of the samples of paint at the National Physical Laboratory from October, 1915, to January, 1917. None of these compounds have a radium content greater than 0·4 mg. per gramme, so that the behaviour of such grades of compound can only be predicted on the assumption that they will obey the same laws as those which have been found to hold for the samples of lower radium content. Further, a series of observations extending over several years would be necessary for the certain prediction of the behaviour of compound for long periods. It has been necessary, however, to make decisions from the results already obtained and the laws deduced from them, without the long delay necessary for experimental confirmation. The conclusions given above are those which experiment has so far shown to be most likely to be justified.

These conclusions are briefly summarised in the two following tables.

CASE A.—*Compound Used in Powder Form.*

Period of enhanced luminosity.	Radium* content.	Luminosity standard.
About six months ...	0·3 mg./gm.	80 p.c. of N.P.L. standard for compound
About one year	0·2 mg./gm.	60 p.c. ,, ,,

* Expressed in gamma ray activity of an equivalent weight of hydrated radium bromide.

Case B.—*Compound Used for Painting Dials.*

Period of enhanced luminosity.	Radium content.	Luminosity standard.
About six months ...	0·6 mg./gm.	140 p.c. of N.P.L. standard for compound
About one year	0·4 mg./gm.	100 p.c. ,, ,,
Sensibly uniform luminosity with a final value about 30 p.c. below that of the other grades	0·2 mg./gm.	60 p.c. ,, ,,

The authors desire to express their acknowledgments to Capt. Creagh-Osborne, Admiralty Superintendent of Compasses, and his assistant, Lieut. Bray. They also wish to express their warm appreciation of the co-operation of manufacturers and others in the work which has been done, and particularly that of Mr. F. H. Glew and Mr. H. A. Hughes.

APPENDIX A.

ABSORPTION OF γ-RAYS BY THE ZINC SULPHIDE IN RADIUM LUMINOUS COMPOUND.

(*a*) *Approximate Graphical Method.*—The first step is to find the difference between the emergent intensities for axial concentration and for uniform distribution of radium in the case of a vessel of square section.

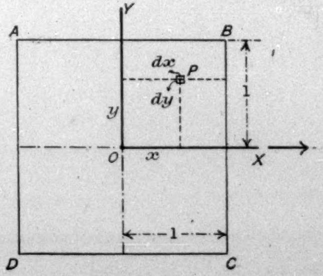


Fig. 13.

Let $A\ B\ C\ D$ be a section of the vessel of side $2l$, 0 being the axis. Then, for axial concentration the intensity in the direction of the OX axis is

$$Ie^{-\lambda l}=I\left\{1-\lambda l+\frac{\lambda^2 l^2}{2!}-\ .\ .\ .\ .\ .\ .\right\}$$

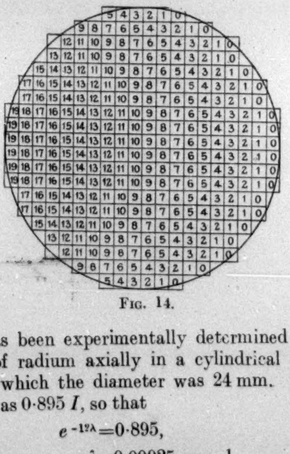
For uniform concentration, however, consider the elementary particle P. The intensity due to this particle P of area $dxdy$ is

$$\frac{dxdy}{4l^2}\,.\,I\,.\,e^{-\lambda(l-x)}.$$

Hence, the total intensity for the whole section is—

$$\left(\frac{I}{4l^2}\right)\int_{-l}^{+l}\int_{-l}^{+l}e^{-\lambda(l-x)}dxdy$$
$$=\frac{I}{2\lambda l}(1-e^{-2\lambda l}),$$
$$=I\{1-\lambda l+\tfrac{2}{3}\lambda^2 l^2-\ .\ .\ .\ .\ .\ .\ \}$$

It will be seen that the two expressions obtained above are equal to 0·1 per cent. if $\lambda l<0{\cdot}1$, while if $\lambda l<0{\cdot}01$ the expressions are identical to 10 parts in a million.


5 4 3 2 1 0
9 8 7 6 5 4 3 2 1 0
12 11 10 9 8 7 6 5 4 3 2 1 0
13 12 11 10 9 8 7 6 5 4 3 2 1 0
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
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19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
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19 18 17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
17 16 15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
15 14 13 12 11 10 9 8 7 6 5 4 3 2 1 0
13 12 11 10 9 8 7 6 5 4 3 2 1 0
12 11 10 9 8 7 6 5 4 3 2 1 0
9 8 7 6 5 4 3 2 1 0
5 4 3 2 1 0

FIG. 14.

Now, λ has been experimentally determined by placing a small tube of radium axially in a cylindrical vessel of zinc sulphide, of which the diameter was 24 mm. The intensity thus found was 0·895 I, so that

$$e^{-12\lambda}=0{\cdot}895,$$

giving $$\lambda=0{\cdot}00925\text{ mm}^{-1}.$$

Now, consider the case of a cylinder of radius a, and let this be divided into a number of small vessels of square section of side $2b=a/10$, as shown in Fig. 14.

It is clear that in this manner uniform distribution over the cylinder may be considered as equivalent to that due to concentration at the axes of 156 square vessels of side $a/10$. Thus the intensity in the direction of the arrow due to each vessel marked "0" will be $(I/312)e^{-\lambda b}$, that due to each vessel marked "1" will be $(I/312)e^{-3\lambda b}$, and so on. In this way it is found that for a vessel of 24 mm. diameter, the intensity is $(0{\cdot}910)\ I$, instead of the $(0{\cdot}895)\ I$, due to axial concentration.

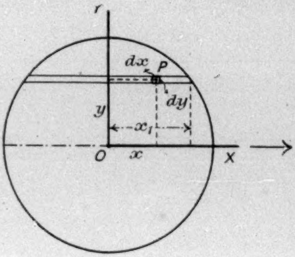


FIG. 15.

(*b*) *Rigid Mathematical Determination.*—Let O be the axis of a cylindrical vessel of radius a, and let P be an elementary particle of area $dx\ dy$. Then the intensity along the OX axis due to the particle at P will be

$$\frac{I}{\pi a^2}dx\ dy\ e^{-\lambda[\sqrt{a^2-y^2}-x]}.$$

Hence, the total intensity is—

$$I'=\frac{I}{\pi a^2}\int_{y=-a}^{y=+a}\int_{x=-\sqrt{a^2-y^2}}^{x=+\sqrt{a^2-y^2}}e^{-\lambda[\sqrt{a^2-y^2}-x]}dx\ dy,$$

$$=\frac{2I}{\lambda\pi a^2}\int_0^a(1-e^{-2\lambda\sqrt{a^2-y^2}})dy.$$

Expanding the exponential and integrating each term this gives—

$$I-\frac{\lambda}{a}\cdot\frac{\pi a^2 I'}{2I}=1-2\lambda a\cdot\frac{\pi}{4}+\frac{4\lambda^2a^2}{2!}\cdot\frac{2}{3}-\ \ldots\ldots$$

$$+(-1)^r\frac{(\lambda a)^r}{r!}\cdot\frac{\Gamma\{(r+2)/2\}}{2\Gamma\{(r+3)/2\}}+\ \ldots\ldots$$

This gives—

$$I'/I=1-\frac{8}{3\pi}(\lambda a)+\tfrac{1}{2}(\lambda^2a^2)-\frac{32}{45\pi}(\lambda^3a^3)+\ \ldots\ .$$

It will be seen that this series is rapidly converging for all values of λa up to unity, and for the sake of convenience the values of the first 12 coefficients are tabulated below. These will enable the value of I'/I to be calculated correct to 0·01 per cent. for all values of λa up to unity.

Table.

The first 12 coefficients in the expression for the γ-ray activity of a sample of radium compound contained in a cylindrical vessel, viz. :—

$$I'/I=z_0+z_1(\lambda a)+z_2(\lambda^2a^2)+\ \ldots\ .$$

Coefficient.	Value.	Coefficient.	Value.
Z_0	+1·000000	Z_7	−0·001642
Z_1	−0·848826	Z_8	+0·000347
Z_2	+0·500000	Z_9	−0·000067
Z_3	−0·226354	Z_{10}	+0·000012
Z_4	+0·083333	Z_{11}	−0·000002
Z_5	−0·025869	Z_{12}	+0·000000
Z_6	+0·006944		

Assuming the value for λ given above $\lambda a=0{\cdot}111$ for $a=12$, so that $I'/I=0{\cdot}9116$, with which the value 0·910 found by the graphical method is in good agreement.

ABSTRACT.

The Paper contains the results of measurements made on various samples of radium luminous compound during the last two years. Determinations of the brightness of the compound in powder form and when made up into paint, and also after the application of the paint to instrument dials, were carried out ; and curves are given showing the rates of decay of luminosity. The radium contents of the compounds were determined by comparison of their γ-ray activities with that of a preparation of pure radium bromide, which is periodically compared with the British radium standard. The various precautions which have to be observed, and the corrections which have to be applied in making the various determinations are explained, and the considerations which should govern the proportion of radium employed for practical purposes are discussed.

DISCUSSION.

Mr. F. H. **Glew** : The difference between some of Marsden's results and those of Dewar and Crookes had been attributed to some difference in the zinc sulphide employed ; he could state that the sulphide supplied to the National Physical Laboratory was of different origin from either of these. Before the war we were almost entirely dependent on foreign sources for the supply of these materials, but we are now quite self-supporting. In

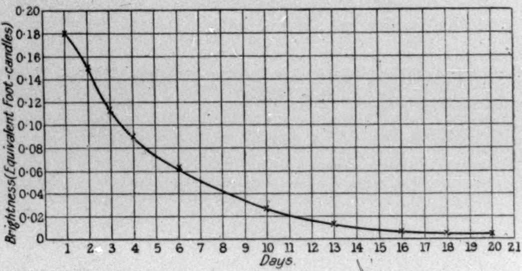
Marsden's work the inside of the tube was completely coated with sulphide, so that the effects were observed by light transmitted through the material. It was better to paint only one half of the tube so that the inner surface, which was exposed to the bombardment, could be directly observed. In the photometer measurements, was there any advantage in placing the coloured filter between the light and the reflecting surface? He preferred to have it in the eye-piece. Earlier records of the coloration due to absorption would be of value. In testing sulphide for γ-ray activity he placed it in contact with the ionisation chamber. Less material was then required for the test, which was an advantage. The authors state only the diameters of the tubes employed; but he had found that there were differences in the results obtained with tubes horizontal or vertical. The length of the tube, therefore, appeared to be an important factor. Further, the results given by a small quantity of sulphide in a large tube would be too high; for the space above the sulphide is filled with the emanation gas, and the γ radiation from it is not subject to the absorption of the sulphide, as is the radiation from the latter itself. The discrepancy between the observed and calculated values of the absorption was probably important. Was this due to some selective action, the sulphide transforming the γ radiation into radiation of some other type? It was singular that specimens prepared by the dry and wet methods should be so similar in their properties after only 4 days; for in the dry method the emanation and all its products are retained.

Dr. **Makower** stated that by using an electroscope of small capacity good γ-ray measurements could be obtained with much less than 1 milligram of radium.

Dr. H. S. **Allen** asked if the zinc sulphide employed was wholly crystalline or partially amorphous. An investigation of its physical condition would be useful.

Prof. **Clinton** said that the difficulties of these measurements were very great and the authors were to be congratulated on having overcome them so effectively.

Mr. J. S. **Dow** said the Paper contained a great deal of useful information. It was common knowledge that a great deal of work on these luminous paints was being done, and they were fortunate in having these results made public. No doubt after the war even fuller information would be forthcoming. His own experience was generally in accord with a number of important results announced by the authors. For example, he found that freshly-prepared compound of 0·4 composition had a brightness not far removed from that mentioned. He rather doubted whether the absolute brightness could be given with exactitude, though an empirical result could no doubt be reproduced quite closely in the laboratory. He had noticed, for example, that if two patches of zinc sulphide and calcium sulphide were excited by visible light, their brightness might be equal when viewed at a distance of a few inches; but the zinc sulphide (giving a green light) appeared much the brighter as the distance of the observer is increased. There was also the question whether the colour-match obtained by using a blue-green glass with an incandescent lamp was sufficiently close to eliminate the purkinje effect at low illumination. With low luminosity materials he had found that a slight obliquity of gaze affected the comparison. He agreed that the colour required was a green with a distinctly blue component. It was curious, however, that while with bright specimens any colour departure was evident almost any green glass appeared to give a colour match at weak luminosities. He had also found that the colour of freshly-prepared and very bright samples was distinctly more blue in tint than that of old and weak ones. The authors' suggestion that the most satisfactory compound, from the standpoint of luminosity in the later period of life and minimum cost, was the 0·2 % material, was a most important result. He thought this would probably be borne out by later experience. In view of the expense, it was obviously necessary to make the radium content as small as possible. Another point on which little information was forthcoming was the order of luminosity necessary in practice for different purposes. Such consideration would probably also favour the use of 0·2 material in most instances. In his experience, in complete darkness, much brighter material might actually cause irradiation. He would like to ask whether the authors had found that in the weaker compounds the initial rise in brightness was disproportionately great and more prolonged than would be expected from the brighter specimens. This was a feature of some tests of which he had received particulars from the United States, and was in some measure supported by his own experiments. Finally, it might be of interest to refer to a series of observations he had recently made of the brightness of a tube coated internally with zinc sulphide and filled with radium emanation, which Mr. Glew had kindly placed at his disposal for test. The initial brightness of this tube was relatively high—of the order of 0·2 foot-candles ; but in three weeks it fell to 1/60th of its initial value. Naturally such a tube would decay much more rapidly than a compound containing powder. Another feature was the very clear exhibition of the formation of local patches of brightness due to condensation of the emanation, to which Mr. Glew had referred. Patches, constantly changing their position, as bright as four times the value of the main luminosity would make their appearance, and then fade away and reappear in another part of the tube. This effect was marked in the early stages of the life, but much less evident at a later stage, when the general brightness had diminished. It would be interesting to know if such effects were met with in varnish or powder materials. In his personal experience he had met only one case, an exceptionally bright tube which was initially much brighter at one end than the other. A few days after the first test the inequality disappeared and had not been noted since.



Decay of Luminosity of Tube filled with Radium Emanation and coated internally with Zinc Sulphide in the course of three weeks after preparation.

Dr. Levy said he had found the luminosity of different samples of zinc sulphide to vary perceptibly in tint when stimulated by daylight, but had not noticed this with α-ray stimulation. In his experience, if a sample of sulphide gave good results when stimulated by daylight, it would also be good when stimulated by radium. The reverse did not hold, however, as a specimen might be poor with daylight, but good with radium excitation.

Mr. T. Smith thought the coloured filter used in the photometric measurements should have been placed as near the lamp as possible, since such a filter acts, to a certain extent, as a self-luminous source. If the tube containing the compound was not much wider than the aperture in the screen, a considerable loss of light would occur, near the edges of the aperture, due to oblique internal reflection at the walls of the tube. For accurate work it was important to have a uniformly illuminated field, and he suggested the desirability of employing tubes with a flat side. Where the tubes were too small to fill the aperture satisfactorily, an optical system could be used to give an enlarged image of the tube and so avoid refractions at large angles. The absorption of the system could be found experimentally with a larger tube.

Dr. A. Russell laid stress on the practical importance of the Paper. He had at times been asked by manufacturers to measure the visibility of watch faces in the dark. The problem was so difficult that hitherto he had done nothing. The present Paper suggested the required practical tests, and a first draft of a specification for securing a standard material could now be written. Progress would be delayed if they waited till all the associated physical and physiological problems were cleared up.

Dr. S. Russ (in a communication which was read by the Secretary) said the decay curves differed appreciably from those of Marsden for Willemite under intense α-ray bombardment. Did the theory held by the authors as to the mechanism of the action differ from that advanced by Prof. Rutherford to explain Marsden's results? Had the authors any measurements of the rate of decay of efficiency of fluorescent screens under the action of X-rays?

Mr. Walsh, in reply, said that if the coloured filter were employed in front of the eye, as suggested by Mr. Glew, its coefficient of transmission for the light emitted by the compound would require to be determined and introduced in the results as well as the coefficient for white light. This would add to the uncertainties of the determinations. To detect change of colour with age would have been difficult in their experiments because of the continuous decrease in luminosity. They had detected differences in colour in samples sent by different makers; but were unaware whether these were from the same source, or made in the same way. The authors had no data on the relation between daylight and radium excitation, and Dr. Levy's remarks were of great interest. The order of accuracy (about one per cent.) was not found to be appreciably affected by altering the position of the coloured screen. The diameter of the smallest tubes usually employed was not less than about twice the apparent width of the aperture. It was not convenient to use a smaller aperture, and they had to do their best with the tubes in which the material was sent. They had not found any samples of radium compound which gave the slow initial rise of luminosity mentioned by Mr. Dow. It is possible, however, that if some radio-active agent other than radium were employed, such, for example, as mesothorium a slow rise of luminosity might very likely result. They would be interested to have further details of the samples referred to by Mr. Dow in this connection.

Mr. Higgins, replying to Dr. Makower, said that they had to employ the radium standards which were available, and these were not suitable for use with a small capacity electroscope. In reply to Mr. Glew, the tube should not be close to the electroscope. If it was 30 to 40 cm. off, the orientation of the tube was of little importance, and small corrections could be applied. The errors introduced by having it too close were similar in character to those produced if an extended source of light is used too near a photometer head. They suspected selective action to be at the root of the discrepancy in the absorptions, but could not definitely say. To investigate the physical properties of the sulphide would be a very long task, and they had not attempted it in the present research.

XIX. *The Resistance to the Motion of a Lamina, Cylinder or Sphere in a Rarefied Gas.* *By* F. J. W. WHIPPLE.

RECEIVED APRIL 13, 1917.

IN Dr. P. E. Shaw's Paper* on the measurement of the pressure in a high vacuum by observation of the logarithmic decrement of the motion of a suspended system, use is made of a formula, due to Prof. Poynting, for the resistance experienced by a plane lamina moving through a rarefied gas. Dr. Shaw mentions that Prof. Poynting had proposed the formula as indicating the order of magnitude of the resistance, and that a numerical factor was possibly to be inserted. A closer examination of the problem and of the validity of its adaptation to the case of a moving cylinder appears to be desirable.

In the rarefied gas with which Dr. Shaw is concerned the free paths of the particles are long compared with the diameter of the containing vessel, so that the velocities are distributed at random—*i.e.*, the velocities of the particles which a lamina meets in any small interval of time are not affected by concussion with particles which have been reflected from it already. In the following investigation it is assumed that the velocities of the particles are distributed according to the Maxwellian law, though it may be questioned whether that law holds good in a gas when collisions with the boundary are more frequent than collisions amongst the particles.

The specification of the conditions of the impact between a particle and the lamina presents special difficulties. Dr. Shaw's argument assumes that the gaseous particle behaves like a sphere rebounding from a perfectly elastic plate. Actually we have to deal with the movement of a molecule which comes into contact with one or more of the molecules of the plate, which molecules are themselves in movement, and there is an interchange of energy.

In this connection reference may be made to a series of papers by M. Knudsen, of Copenhagen. The problems discussed by Knudsen are connected with diffusion and viscosity, and he finds that the assumption † that the directions of motion of the gas particles rebounding from a solid boundary are distributed at random and independent of the directions in which they approach it is consistent with the experimental evidence. The velocities of the rebounding particles depend only on the temperature of the boundary.*

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* "Proc." Phys. Soc., 1916, p. 171.

† "Ann. d. Physik," XXVIII., 1, pp. 75-130. I am indebted to Dr. Chapman for this reference.

The corresponding assumption in the case of a moving solid is that the velocities of the rebounding particles *relative to the solid* are determined by the temperature of the solid, and are not affected by its motion. The molecules of the gas may be thought of as absorbed and emitted again, and the pressure as made up to two parts, corresponding with the impulses which reduce the molecules to relative rest and to the impulses which drive them outwards from the surface.

Resistance to the Motion of a Lamina.

If ρ_0 be the density of the gas as a whole and V the standard speed of the molecules, the density of those having speeds between u and $u+du$, and moving in directions with polar co-ordinates from θ to $\theta+d\theta$ and azimuths φ to $\varphi+d\varphi$ is

$$d\rho=\rho_0\frac{\sin\theta d\theta d\varphi}{4\pi}\sqrt{\frac{54}{\pi}}e^{-3u^2/2V^2}\frac{u^2du}{V^3}.$$

We consider in the first instance a lamina moving at right angles to its plane with speed v and measure θ from the direction of motion of the lamina.

The relative velocity with which a particle approaches the lamina has a normal component $u\cos\theta+v$. The total mass of the particles with velocities within the prescribed limits and striking unit area of the lamina in unit time is $(u\cos\theta+v)d\rho$. The corresponding pressure is $dp=(u\cos\theta+v)^2d\rho$.

The particles hit the positive side of the lamina—*i.e.*, the side to which it is moving—provided that $u\cos\theta+v$ is positive. The limiting value of θ is a given by $\cos a=-v/u$.

Evidently a is a little greater than a right angle. The total pressure due to the impacts on the positive side of the lamina is p_1 given by

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* It should be mentioned, however, that in a later Paper ("A.d.P.," XXXIV., 4) Knudsen finds it necessary to introduce a "coefficient of accommodation," which permits the hypothesis that the energy of the rebounding particles depends in part on the energy with which they approach the boundary. The introduction of such a coefficient would increase slightly the reactions found in the present Paper. The hypothesis of Smoluchowski ("Contributions to the Theory of Transpiration, &c., in Rarefied Gases," "Bull." International de l'Acad. des Sciences de Cracovié, 1910, A. No. 7, p. 295) is similar to Knudsen's. Cf. Soddy & Berry, Roy. Soc. "Proc." 1910, A., Vol. LXXXIV., p. 584.

$$p_1=\frac{\rho_0}{4\pi}\int_0^\infty\int_0^{2\pi}\int_0^\alpha (u\cos\theta+v)^2\sin\theta d\theta d\phi\sqrt{\frac{54}{\pi}}e^{-3u^2/2V^2}\frac{u\,du}{V^3}$$
$$=\sqrt{\frac{27}{2\pi}}\rho_0\int_0^\infty\frac{(u+v)^3}{3}e^{-3u^2/2V^2}\frac{udu}{V^3}.$$

The corresponding pressure p_2 on the back of the lamina can be written down in the same way. No particle whose velocity is less than v will meet the lamina on this side so that

$$p_2=\sqrt{\frac{27}{2\pi}}\rho_0\int_v^\infty\frac{(u-v)^3}{3}e^{-3u^2/2V^2}\frac{udu}{V^3}$$
$$=\sqrt{\frac{27}{2\pi}}\rho_0\int_0^\infty\frac{(u-v)^3}{3}e^{-3u^2/2V^2}\frac{udu}{V^3}+\varepsilon_2,$$

where

$$\varepsilon_2=\sqrt{\frac{27}{2\pi}}\rho_0\int_0^v\frac{(v-u)^3}{3}e^{-3u^2/2V^2}u\frac{du}{V^2},$$
$$<\sqrt{\frac{27}{2\pi}}\rho_0\int_0^v\frac{v^4}{3}\frac{du}{V^3}\quad\text{or}\quad\sqrt{\frac{27}{2\pi}}\rho_0\frac{v^5}{15V^3}.$$

The resultant difference of pressure on the two faces of the lamina is—

$$p_1-p_2=2\sqrt{\frac{27}{2\pi}}\rho_0\int_0^\infty\left(u^2v+\frac{v^3}{3}\right)e^{-3u^2/2V^2}u\frac{du}{V^3}-\varepsilon_2,$$
$$=2\sqrt{\frac{2}{3\pi}}\rho_0\left(vV+\frac{v^3}{2V}\right)-\varepsilon_2.$$

As v is small compared with V, the terms in v^3 and v^5 may be ignored, and the effective pressure corresponding with the impulses which reduce the particles to relative rest is therefore

$$2\sqrt{\frac{2}{3\pi}}\rho_0vV.$$

To find the pressure corresponding with the impulses which set the particles in motion again we require the difference between the masses emitted from unit areas on the two sides of the lamina in unit time and the mean relative velocity normal to the surface.

The statement that the difference in question is equal to $v\rho_0$, the mass of gas which would be swept up by the lamina in unit time if the particles were not in motion is plausible, and is borne out by integration of expressions similar to those in the preceding paragraphs.

To determine the mean relative velocity normal to the surface we assume that the law of distribution of relative velocity is the same as in the case of the fixed boundary of an enclosure containing gas at uniform temperature. In that case the distribution of velocities of the particles leaving the boundaries is such as to maintain the Maxwellian distribution throughout the gas.

In such a case the mass striking unit area of the wall in unit time is—

$$\iiint u\cos\theta \, . \, d\rho = \frac{V\rho_0}{\sqrt{(6\pi)}},$$

and the impulse given to this mass as it leaves the surface is—

$$\iiint (u\cos\theta)^2 d\rho = \frac{V^2\rho_0}{6},$$

so that the mean normal velocity with which a particle leaves the surface is $\sqrt{\frac{\pi}{6}}V$.

Accordingly in the problem of the moving lamina the resultant pressure due to the impulses driving the particles outwards from the surfaces is

$$\sqrt{\frac{\pi}{6}}Vv\rho_0.$$

It has been shown already that the pressure due to the impulses which reduced the particles to relative rest was

$$2\sqrt{\frac{2}{3\pi}}Vv\rho_0.$$

The effective force per unit area is therefore

$$\left(\sqrt{\frac{\pi}{6}}+2\sqrt{\frac{2}{3\pi}}\right)Vv\rho_0,$$

or

$$\frac{4+\pi}{\sqrt{(6\pi)}}Vv\rho_0.$$

The pressure which the gas exerts on the boundary is p_0, given by

$$p_0 = \tfrac{1}{3}\rho_0 V^2,$$

so that the effective force on the lamina is

$$(4+\pi)\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0.$$

It will be seen that if the lamina had been assumed to be perfectly elastic * the factor $4+\pi$ would have been replaced by 8.

Turning now to the problem of the motion of a lamina in any direction, we may resolve its velocity into components v perpendicular to its surface and w parallel thereto.

In specifying the direction from which a particle is approaching the lamina, we take θ as measured from the direction of v and the azimuth φ as measured from the plane containing v and w. If u be the speed of a molecule, the component of the relative velocity in the direction opposite to w is

$$u \sin\theta \cos\varphi + w.$$

R_1, the resultant stress parallel to the plane caused by impacts on its positive side, is given by

$$R_1 = \frac{\rho_0}{4\pi}\int_0^\infty\int_0^{2\pi}\int_0^\alpha (u\cos\theta+v)(u\sin\theta\cos\varphi+w)\sin\theta\,d\theta\,d\varphi \sqrt{\frac{54}{\pi}}\,e^{-3u^2/2V^2}\frac{u^2du}{V^3}.$$

On integration with respect to φ the term independent of w vanishes, and—

$$R_1 = \frac{\rho_0 w}{4}\int_0^\infty \frac{(u+v)^2u}{V^3}\sqrt{\frac{54}{\pi}}\,e^{-3u^2/2V^2}\,du.$$

Dealing with the corresponding stress on the negative side in the same way and disregarding the terms of lower order as before we find that the effective traction is

$$\frac{\rho_0 w}{2}\sqrt{\frac{54}{\pi}}\int_0^\infty e^{-3u^2/2V^2}\frac{u^3du}{V^3},$$

i.e., $$\sqrt{\frac{2}{3\pi}}\rho_0 wV, \quad \text{or} \quad 2\sqrt{\frac{3}{2\pi}}\frac{w}{V}p_0.$$

The velocities of the particles leaving the surface are distributed symmetrically round the normal, so that there is no corresponding tangential force. Accordingly, it follows that if a lamina is moving with such a velocity that the components at right angles to its surface and parallel to the surface are v, w respectively, the corresponding components of the resistance to its motion are

$$(4+\pi)\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0, \quad 2\sqrt{\frac{3}{2\pi}}\frac{w}{V}p_0 \text{ respectively.}$$

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* As in the discussion on Dr. Shaw's Paper, *l.c.* p. 175.

Resistance to the Motion of a Cylinder.

We proceed to the case of a cylinder moving in a direction at right angles to its length. The cylinder is regarded as so long that the tractions due to the molecules of the gas on its ends may be ignored in comparison with the effects on the cylindrical surface.

The pressures and tractions on portions of the cylinder with parallel tangent planes may be combined as in the case of the lamina. The velocity of the cylinder can be resolved into components perpendicular to and parallel to any tangent-plane and the corresponding pressure and traction can then be resolved in the direction of motion. If D be the diameter of the cylinder and L its length, the total resistance it encounters when moving with speed v is seen to be

$$\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0LD\int_0^\pi[(4+\pi)\cos^2\psi+2\sin^2\psi]\frac{d\psi}{2},$$

or,

$$\frac{6+\pi}{4}\sqrt{\frac{3\pi}{2}}\cdot\frac{v}{V}p_0LD.$$

In Dr. Shaw's Paper it was assumed that the resistance to the motion of a cylinder would be the same as the resistance to the motion of a lamina occupying the central plane of the cylinder. On the present hypothesis this is very nearly the case. The cylinder encounters more particles but, on the other hand, the impacts are not direct. The ratio of the resistance experienced by the cylinder and by the lamina of area LD is—

$$\frac{\pi(6+\pi)}{4(4+\pi)} \quad \text{or} \quad 1{\cdot}005.$$

Resistance to the Motion of a Sphere.

The case of the sphere can be treated in the same way as that of the cylinder. If a be the radius of the sphere and v its velocity, the resistance is—

$$\sqrt{\frac{3}{2\pi}}\frac{v}{V}p_0a^2\,.\,2\pi\int_0^{\pi/2}[(4+\pi)\cos^2\psi+2\sin^2\psi]\sin\psi\,d\psi,$$

or

$$(8+\pi)\sqrt{\frac{2\pi}{3}}\frac{v}{V}p_0a^2.$$

The ratio to the resistance experienced by a plane lamina of area πa^2 is—

$$\frac{2(8+\pi)}{3(4+\pi)} \quad \text{or} \quad 1{\cdot}04.$$

Damping of Motion in a Rarefied Gas.

When the formula obtained above for the resistance to the motion of a lamina is substituted for that given by Dr. Shaw, the coefficient K in the equation of motion is given by

$$K=\frac{(4+\pi)}{\sqrt{6\pi}}\,\frac{\rho VSa}{I},$$

and the density of the gas can be calculated from the formula

$$\rho=\frac{4\sqrt{(6\pi)}}{4+\pi}\,\frac{I\lambda}{VSaT},$$

or

$$\rho=2{\cdot}43\,\frac{I\lambda}{VSaT},$$

where S is the total effective area of the moving bodies, a the distance of their centres from the axis of rotation, I their moment of inertia, T the period of oscillation and λ the logarithmic decrement.

The coefficient 2·43 replaces the 3 of Dr. Shaw's Paper, so that the pressures calculated there should be reduced by about 20 per cent.

Added May 12*th*, 1917.

Motion in Connected Vessels.

The standard velocity of the gas-molecules can be eliminated from the resistance-formulæ by use of the equation $V^2=3R\theta_0$, in which R is the gas-constant and θ_0 the absolute temperature.

For example, the formula for the effective pressure on the lamina becomes

$$\frac{(4+\pi)vp_0}{\sqrt{(2\pi R\theta_0)}}.$$

The occurrence of the factors p_0 in the numerator and $\theta_0^{\frac{1}{2}}$ in the denominator is of interest, as it is known* that when two vessels at different temperatures and containing a rarefied gas are connected by a tube of narrow bore the pressures in the two vessels are not equal but related by the relation

$$\frac{p_0}{\theta_0^{\frac{1}{2}}}=\frac{p_1}{\theta_1^{\frac{1}{2}}},$$

p_0, p_1 being the pressures, θ_0, θ_1 the absolute temperatures.

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* Soddy and Berry, *l.c.*, p. 583, quote Osborne Reynolds, "Phil. Trans." 1879, p. 727, and Knudsen, "Annalen der Physik," 1910, Vol. XXXI., p. 205.

Our formula shows that if equal laminæ are moving with equal speeds in two such vessels, the resistances to their motions will be equal. The oscillations of two suspended systems, such as Dr. Shaw's, made with the same dimensions would in like circumstances be damped at the same rate.

ABSTRACT.

The investigation is carried out on the assumptions that the free-paths of the particles of the gas are long compared with the dimensions of the moving body, and that the motion relative to the body of the particles which rebound from it depends only on its temperature. It is shown that if v, w be the components of velocity perpendicular to the surface of a lamina and parallel thereto the corresponding components of the resistance are

$$(4+\pi)\sqrt{\frac{3}{2\pi}}\frac{v}{V}p \quad \text{and} \quad 2\sqrt{\frac{3}{2\pi}}\frac{w}{V}p,$$

where V is the standard (root-mean-square) speed of the gas-particles and p is the gas-pressure.

The resistance to the motion of a cylinder or a sphere is found to differ very slightly from the resistance to a lamina occupying the central section. The formulæ are applicable to the problem of the damping of the oscillations of a system suspended in a rarefied gas.

DISCUSSION.

Mr. G. D. West called attention to the fact that the author's value for the ratio of the normal to the tangential resistance of a lamina, was in good agreement with that previously obtained by Knudsen. This experimenter had also deduced a formula for the resistance of a sphere moving in a rarefied gas, and it would be interesting to see how this result compared with the author's. In connection with the damping of vibratory motion, a large amount of careful work had been done by Hogg.

Dr. D. Owen said that much interest now attached to the study of electrical phenomena occurring in the highest vacua. The present Paper was valuable as assigning a more precise figure for the constant to be used in reducing observations for pressure measurement by the method of damping. It should be noted, however, that the expressions in the Paper were arrived at on a particular assumption as to the nature of the recoil of the gas molecules, the experimental basis of which could not as yet be regarded as assured.

The damping method—which was used as far back as 1880 by Crookes—hardly seemed likely to be adaptable at pressures below 0·01 μ of mercury. It might, on the other hand, be useful in work at pressures well above 1 μ, and it would be of practical value to know to what extent the formulæ given in the Paper would lead to error at higher and higher ratios of mean free path to diameter of vessel. For the lowest vacua Dr. Langmuir ("Phys. Rev.," 1913) has worked out a very promising method. A horizontal disc is driven by means of a rotating magnetic field at any speed up to 10,000 revs. per min. This moving disc produces a static torque on a parallel suspended disc of mica provided with a mirror, the steady deflection of which can then be measured. Pressures down to 10^{-4} μ or less might thus be determined, given the appropriate mathematical constant, which, however, apparently remained to be calculated.

Dr. P. E. Shaw (communicated remarks): Mr. Whipple's results obtained by the application of exact analysis will, I think, prove very useful. Each of the three cases (lamina, cylinder and sphere) is applicable to well-known special high vacuum apparatus. The pressure existing at any time in a sealed-off high vacuum cannot be found by the McLeod gauge ; the use of the electric discharge method is very rough, and may introduce ionisation, de-occlusion and other troubles. Other methods, no doubt, are possible ; but when the apparatus has a system moving with pendular or torsional vibration the log. dec. method is simple in the extreme, and requires no accessory plant. The operator need not go near the apparatus, for a mirror and telescope will show the amplitude of vibration. In the gravitation apparatus I used the amplitude could be detected accurately to an angle of 2 secs. The damping, in one case, for a vacuum of about 1/1,000,000 atmosphere press. was about 3·4 secs. of angle This shows the sensitiveness obtainable. For special cases the method would prove convenient, and I think Mr. Whipple's investigation will prove useful to the experimentalist. The 20 per cent. reduction in the values of the pressure, as found by Mr. Whipple, does not surprise me ; for, as stated, I only aimed at approximate values.

XX. *An Alternating Current Bridge Method of Comparing Two Fixed Inductances at Commercial Frequencies.* By T. PARNELL, *M.A., Lecturer in Physics, University of Queensland.*

COMMUNICATED BY PROF. T. R. LYLE.
RECEIVED MAY 4, 1917.

THE simple Maxwell inductance bridge cannot be used conveniently for the comparison of fixed inductances on account of the tedious double balance necessary, though by reason of its simplicity and symmetry the bridge has much to recommend it.

By making use of a current detector whose deflections depend on the component of the current in quadrature with the E.M F. applied to the bridge, it is possible to make the condition for no deflection depend either chiefly on the inductances or chiefly on the resistances.

The arrangement of the bridge is shown in Fig. 1. In series with the bridge is placed either a non-inductive resistance, *R*, or a condenser, *K*. If *R*, *r* and *s* are all large compared with p, q, g, $P\omega$ and $Q\omega$, it will be shown that, with *R* in circuit, the balance depends chiefly on the inductances *P* and *Q*, and that with *K* in circuit the balance depends chiefly on the resistances.

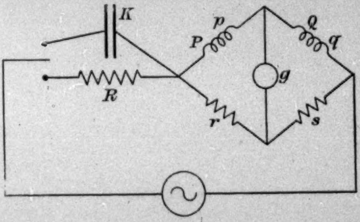

FIG. 1.

With *R* in circuit, *s* is adjusted for no deflection; this gives an approximate balance for the inductances. Then, with *K* in circuit, *q* is adjusted for no deflection; this gives an approximate balance for the resistances. The process is repeated, and after a few balances values of *s* and *q* are obtained that give a balance with either *R* or *K* in circuit, in which case the bridge is completely balanced, and

$$\frac{P}{Q}=\frac{p}{q}=\frac{r}{s}.$$

The first few balances need only be made roughly, and the whole process can be carried out very quickly; when the apparatus is adjusted, an unknown inductance can be balanced against a standard to one part in a few thousand in a few minutes.

As detector a sensitive moving-coil galvanometer and rotating commutator may be used. In this case the commutator is mounted on the axle of the small alternator or rotary converter that supplies the bridge current, and the brushes of the commutator are adjusted till no deflection is produced when the E.M.F. used on the bridge is applied through the commutator to a high non-inductive resistance in series with the shunted galvanometer, thus showing that the commutator is in quadrature with the E.M.F. When the commutator is thus adjusted the deflection of the galvanometer is approximately proportional to the component of the current in quadrature with the E.M.F. applied to the bridge.

A more convenient detector is the Sumpner reflecting electrodynamometer, manufactured by Paul. This instrument is fully described in a Paper by Sumpner and Phillips.* The suspended coil moves in the field of a laminated iron electromagnet excited by a field winding with a large number of turns; the resistance of this field winding is small compared with its reactance, and therefore the total flux through the field coils is approximately in quadrature with the applied E.M.F. The flux density is sensibly constant throughout the air-gap and proportional to the total flux. When, therefore, the moving coil is connected across the bridge, and an E.M.F. in phase with the E.M.F. used for the bridge is applied to the field coils, the mean torque on the moving coil is approximately proportional to the component of the current in quadrature with the bridge E.M.F.

If the mutual inductances between the various conductors, and also their capacities and the inductances of *R*, *r* and *s*, are negligible, the final balance is independent of wave-form and frequency. The preliminary balances depend on wave-form and frequency. An approximate knowledge only of the conditions for these preliminary balances is required; an approximate calculation of the current through the detector, therefore, is sufficient, and it may be assumed with a sufficient degree of accuracy that the mean torque on the moving coil is proportional to the component of the current in quadrature with the bridge E.M.F.

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* "Proc." Phys. Soc., Vol. XXII., p. 395. "Phil. Mag.," August, 1910, p. 309.

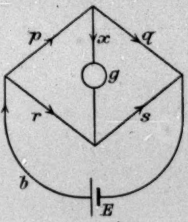

FIG. 2.

Consider the bridge as in Fig. 2

For continuous currents $x=\frac{qr-ps}{\Delta}\,.\,E$,

where $\Delta=bg(p+q+r+s)+b(p+r)(q+s)+g(p+q)(r+s)$
$\qquad+pq(r+s)+rs(p+q)$.

For the bridge, Fig. 3, with the harmonic E.M.F., $e=E\sin\omega t$,

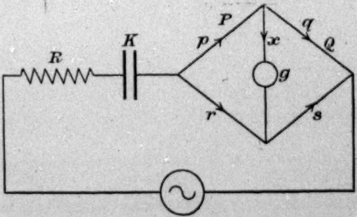

FIG. 3.

if the inductance of g is small compared with P and Q,

p becomes the operator $p+i\omega P$
q " " $q+i\omega Q$
b " " $R-\frac{i}{\omega K}$,

and if R, r, s are large compared with p, q, g, $P\omega$ and $Q\omega$, Δ becomes

$$(p+i\omega P)\left(r+R-\frac{i}{\omega K}\right)s+(q+i\omega Q)\left(s+R-\frac{i}{\omega K}\right)r$$

$$+\left(R-\frac{i}{\omega K}\right)rs \text{ approximately}$$

$$=M+iN,$$

where $$M=p(rs+Rs)+q(rs+Rr)+Rrs+\frac{1}{K}(sP+rQ)$$

$$M=\omega P(rs+Rs)+\omega Q\,(rs+Rr)-\frac{1}{\omega K}rs.$$

$$\therefore \quad x=\frac{(qr-ps)+i(\omega Qr-\omega Ps)}{M+iN}\,.\,E.$$

If $A=\frac{q}{p}-\frac{s}{r}$, *i.e.* the "resistance error," and $B=\frac{Q}{P}-\frac{s}{r}$, *i.e.* the "inductance error,"

$$x=\frac{prA+i\omega PrB}{M+iN}\,.\,E$$

$$=\frac{prMA+\omega PrNB+i(\omega PrMB-prNA)}{M^2+N^2}E.$$

The mean torque on the moving coil is proportional to the component of x in quadrature with E, *i.e.* to

$$\frac{\omega PrMB-prNA}{M^2+N^2}.$$

$\therefore$ Mean torque on moving coil $=XB-YA$

where $$\frac{X}{Y}=\frac{\omega PM}{pN}$$

$$=\frac{\omega P\left\{p(rs+Rs)+q(rs+Rr)+Rrs+\frac{1}{K}(sP+rQ)\right\}}{p\left\{\omega P(rs+Rs)+\omega Q\,(rs+Rr)-\frac{1}{\omega K}.\,rs\right\}}.$$

Consider now the two cases for preliminary balances :—

Case I. (inductive)—

$$\frac{1}{K}=0$$

$$\frac{X}{Y}=\frac{\omega PRrs}{\omega p\,\{P(rs+Rs)+Q(rs+Rr)\}} \text{ approx.}$$

$$=\frac{Rrs}{prs+pRs+\frac{Q}{P}(prs+pRr)}.$$

$\therefore$ $\frac{X}{Y}$ is positive and fairly large.

Case II. (resistance)—

$$R=0$$

$$\frac{X}{Y}=\frac{\omega P\left\{(p+q)rs+\frac{1}{K}(sP+rQ)\right\}}{p\left\{rs\omega(P+Q)-rs\frac{1}{\omega K}\right\}}$$

$$=-\frac{\omega^2 P\left\{(p+q)rs+\frac{1}{K}(sP+rQ)\right\}}{\frac{1}{K}prs} \text{ approx.}$$

$\therefore \frac{X}{Y}$ is negative, and small for ordinary values of P, Q and ω.

The above assumes an harmonic E.M.F., but since the final balance is independent of frequency and wave-form, the effect of the higher harmonics in the case of the E.M.F. obtained from an ordinary machine may be neglected, for it is clear from the above expressions that the effect of the higher harmonics in the preliminary balances will be much less than that of the fundamental.

A more rigorous treatment leads to the same result as the simpler theory given above.

For no deflection the mean torque is zero—

$$i.e. \qquad XB-YA=0\,;$$

or

$$X\left(\frac{Q}{P}-\frac{s}{r}\right)-Y\left(\frac{q}{p}-\frac{s}{r}\right)=0.$$

If this preliminary balance is obtained by varying s with R alone in circuit (Case I.), X and Y are of the same sign, and X is fairly large compared with Y; hence the balance depends chiefly on the inductances. If in this balance q is too small for a resistance balance A is negative, and therefore B is negative, and the value of s is too great. Similarly if the value of q is too great, the value of s obtained will be too small.

If a balance be now obtained by varying q (or p) with K alone in circuit (Case II.), X and Y are of opposite sign and X is fairly small compared with Y; hence the balance depends chiefly on the resistances. If in the first balance q is too small, a value of s too great for a true inductive balance is obtained—*i.e.*, in this balance B is negative, and therefore A is positive, and the value of q is too great. The next inductive balance will, therefore, give too small a value of s, which means that the next resistance balance will give too small a value of q, and so on.

Two series of values of s and q respectively are thus obtained whose terms are alternately too great and too small, and these terms rapidly converge to final values corresponding to a balance of both inductances and resistances—

i.e.
$$\frac{P}{Q}=\frac{r}{s}=\frac{p}{q}.$$

The following experimental results serve to illustrate the use of the method and to show the agreement of the above theory with experiment.

In these measurements P was a standard inductance and Q the coil to be measured. An inductive balance was first obtained by varying s, then a resistance balance by adding resistance to the arm containing Q, or to the arm containing P, by means of a dial box and graduated non-inductive rheostat, which could be switched into either arm.

Column I. gives the series of values of s for inductive balances; Column II. the corresponding values of the added resistances.

One example of the use of the rotating commutator is given; the other measurements were made with the electrodynamometer as detector.

Coil A.—Resistance, 15·6 ohms; P=100 M.H., R=2,000 ohms, $K=\frac{1}{2}$ M.F., r=2,000 ohms, p=34·8 ohms

Using electrodynamometer.		Using rotating commutator.	
I. s.	II. Resistance added to arm Q.	I. s.	II. Resistance added to arm Q.
	0		0
3,480·0		3,300·0	
	45·0		43·0
3,040·0		3,060·0	
	37·0		37·0
3,120·0		3,108·0	
	38·7		38·22
3,099·0		3,101·0	
	38·39		38·04
3,102·6		3,102·0	
	38·425		38·06
3,102·3		3,101·7	
	38·42		...
3,102·4		...	
	...		
Q=155·12 M.H.		Q=155·08 M.H.	

Coil B.—Resistance, 0·73 ohm ; $P=10$ M.H., $R=2{,}000$ ohms, $K=\frac{1}{2}$ M.F., $r=2{,}000$ ohms, $p=7\cdot4$ ohms.

I. *s*.	II. Resistance added to arm *Q*.
	0
1,000·0	
	3·0
800·0	
	2·2
860·0	
	2·45
846·0	
	2·36
853·0	
	2·405
850·0	
	2·39
850·1	

$Q=4\cdot250$ M.H.

Coil C.—Resistance, 0·3ohm ; $P=10$ M.H., $R=2{,}000$ ohms, $K=\frac{1}{2}$ M.F., $r=2{,}000$ ohms, $p=7\cdot4$ ohms.

I. *s*.	II. Resistance added to arm *Q*.
	0
129·0	
	0·13
119·0	
	0·09
121·4	
	0·105
120·0	
	0·101
120·6	
	0·102
120·5	

$Q=0\cdot602$ M.H.

With the larger inductances the final balance could easily be made to 1 in 10,000, but this degree of accuracy cannot be expected in the actual results, as the non-inductive resistance boxes were specified as correct to 1 in 1,000 only and no small corrections were made.

Measurements of a number of coils with various values of the voltage and of R, K and r gave results consistent within 1 in 1,000. As a further test of the accuracy of the method, two coils were placed a considerable distance apart, and their inductances (including the leads) measured in situ, singly and in series. The results of two such measurements are given below.

Inductance of A	155·12 M.H.
Inductance of A'	158·02 M.H.
Sum of inductances	313·14 M.H.
Inductance of A and A' in series	313·24 M.H.

Inductance of B	4·250 M.H.
Inductance of B'	3·942 M.H.
Sum of inductances	8·192 M.H.
Inductance of B and B' in series	8·190 M.H.

Reversing the connections of A' or B' did not appreciably affect the final balances.

Owing to the presence of R or K, it is necessary to use fairly high voltages in order to keep up the sensitiveness. The value of R need not be known with much accuracy, so that any reasonably non-inductive resistance may be used, and as and s are large compared with p and q, the voltage is limited only by the permissible current through the inductance coils. In the measurements made, voltages from 10 to 85 were used, and the higher voltages were found the more satisfactory

The mutual induction between R and the rest of the bridge is liable to produce considerable error; it is necessary, therefore, to keep R some little distance away.

Leakage to earth and electrostatic capacity effects gave some trouble. The detector was kept permanently in circuit, and the alternating-current supply connected to the two terminals of the bridge through a pair of tapping keys; the apparatus was manipulated until putting down either key alone gave only a negligible deflection. In obtaining a balance the two keys were put down simultaneously. The "false zero" obtained by keeping the detector in circuit fluctuates to some extent, but little difficulty was experienced from this cause, as it was always sufficiently steady to enable the final balance to be made accurately.

When the rotating commutator was used it was found that the best results were obtained by taking the bridge current from a small transformer, with secondary well insulated and shielded from primary connected to the machine. The galvanometer was connected to the brushes bearing on the split part of the commutator, and the bridge to the brushes on the rings. When the core of the transformer, the frame of the machine and one of the rings of the commutator were earthed, the stray effects due to leakage, capacity, &c., were reduced to a negligible amount.

In setting the brushes of the commutator the current from the transformer was passed through the commutator and the suitably shunted galvanometer in series with R, and the brushes adjusted for no deflection. This allowed for any change of phase occurring when the bridge current was taken from the transformer. It was found that great accuracy in the position of the brushes was not necessary. If the setting was not sufficiently exact, the defect was noticed at once, as the values of s and q did not alternate in the manner described above.

The use of the rotating commutator was found much more troublesome than that of the electrodynamometer, and the results obtained were somewhat less consistent.

When the electrodynamometer was used as detector, the currents for the bridge and for the field were taken from a small transformer having two secondaries well insulated and shielded from one another. It was found that with this arrangement electrostatic and capacity effects could be reduced to a negligible amount by earthing the core of the transformer and one terminal of the moving coil. Voltages from 40 to 75 volts were employed on the field coils.

In conclusion I wish to express my thanks to Prof. Lyle for his kindness in reading through this Paper and offering valuable criticisms. It was at his suggestion that the simpler proof given here was substituted for a more elaborate and cumbersome one employed originally.

NOTE.—Sumpner and Phillips mention the use of the electrodynamometer for the direct comparison of inductances, but they state that the resistance balance is unnecessary This does not agree at all with the results obtained in this Paper.

ABSTRACT.

The Paper describes a method of avoiding the troublesome double adjustment required in Maxwell's method of comparing inductances. A current detector, whose deflections depend on the component of the current in quadrature with the E.M.F., is employed, which makes it possible to arrange that the condition for no deflection depends chiefly either on the inductances or resistances. In series with the bridge is placed either a non-inductive resistance or a capacity. In the first case the balance depends chiefly on the inductances, and in the second case on the resistances. A few alternate repetitions of the two adjustments suffice to balance the bridge, both for resistances and inductances. As detector a sensitive moving coil galvanometer

in conjunction with a commutator, or a Sumpner electrodynamometer, may be employed; the latter proved more satisfactory.

DISCUSSION.

Dr. D. **Owen** regarded the Paper as a valuable contribution to the subject of alternate-current bridge measurements. To secure zero current through the detector of a bridge to which an alternating voltage is applied it is necessary to satisfy two conditions. It has been claimed that by the use of the alternate-current moving-coil galvanometer inductances or capacities might be accurately determined without the necessity of a double adjustment. The claim is a mistaken one. The mere attainment of zero deflection, implying zero torque on the moving coil, by itself cannot give the information sought, for neither of the conditions for zero current in the detector is necessarily satisfied. The author of the present Paper shows a way out of the difficulty—namely, by applying alternately voltages to the bridge which are approximately—but not necessarily exactly—in quadrature with each other, the voltage applied to the field coil remaining unchanged. The attainment of a balance in *both* of these cases justifies the inference that the galvanometer current due to the voltage impressed on the bridge is zero, and that the two fundamental conditions of balance are simultaneously reached. A Paper by Weibel in the Bureau of Standards "Bulletin" (February, 1917) deals very fully with this same problem. Weibel points out how the tests may be made also by varying the phase of the voltage applied to the field coil, voltages in quadrature being secured by the aid of a phase-shifting device. The present author applies his method solely to the Maxwell bridge for comparing two self-inductions. There seems, however, no good reason for this exclusive treatment. The mode of procedure advocated is applicable to all the bridge methods, and those methods in which the two conditions of balance are independently attainable will still have the advantage in practice over those, such as the Maxwell bridge of the Paper, in which the opposite is the case.

Mr. T. **Smith** pointed out that in double adjustments of this nature, if the wrong adjustment were made first, the successive errors got larger instead of smaller. It would have been helpful if the author had extended his mathematics somewhat, and given some criterion (*e.g.*, some function of the various quantities involved being either positive or negative) as to which of the adjustments—inductance or resistance—should be made first.

Dr. **Bryan** asked if the method could be used for small inductances of a few micro-henries?

Prof. **Boys** asked if inductances with iron cores could be measured, or only air cored coils, in which the sine law was nearly fulfilled?

Dr. **Sumpner** said, in reply, that the author had not gone in great detail into the range of usefulness of the method. He might point out, however, that the sine law was not essential to work with the electrodynamometer. The author did not appear to have experienced any difficulty about the order of the adjustments; though in some cases this might doubtless cause considerable trouble. He admitted the validity of the author's criticism of the Paper by himself and Phillips; and agreed that the double adjustment was essential. At that time, however, he was principally concerned with describing the instrument, and was not using small inductances in which the resistance effect was important.

Mr. A. **Campbell** communicated the following: Mr. Parnell's system is interesting, and may be of value in other methods where two adjustments are necessary to obtain a balance. In connection with his commutator method it may be mentioned that L. T. Robinson applied a commutator in the galvanometer circuit in a somewhat similar way for testing transformers by inductance methods ("Trans." American Institute of Electrical Engineers, p. 1040, Vol. XXVIII., June 30, 1909).

XXI. *On the Wave-lengths and Radiation of Loaded Antennæ.*

By BALTH. VAN DER POL, JUN., *D.Sc.* (*Utrecht*).

COMMUNICATED BY PROF. FLEMING. RECEIVED APRIL 13, 1917.

THE real function of an antenna as used in wireless telegraphy is to radiate electromagnetic waves into space. The actual measurement of the amount of this radiation is a difficult matter, though some suggestions in this direction have been made by Erskine-Murray * and De Groot.† To reach a high degree of exactness in such measurements, however, seems to be hardly possible as yet, so that a general theoretical investigation may have some interest.

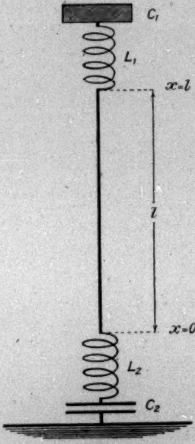

FIG. 1.

Antennæ in practice have a great variety of forms. They often consist of one or more straight, parallel, vertical wires continued at the top by others more or less horizontal. At the bottom of the vertical part an inductance-coil is placed, used as one side of a mutual inductance arrangement for the purpose of transferring electromagnetic energy from a primary oscillation circuit to the radiator.

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* "Jahrbuch der Drahtl. Telegr. und Teleph.," V., 499, 1912.

† "Jahrbuch der Drahtl. Telegr. und Teleph.," VIII., 109, 1914.

A condenser of variable capacity is often inserted at the earthed base of the aerial, especially when the latter is used as a receiving arrangement. The horizontal wires at the top of a symmetrical antenna act principally as a capacity attached to the vertical part of the aerial.

To bring the many forms of these arrangements within the reach of mathematical analysis we assume that the antenna has the form of Fig 1.—*i.e.*, there is a long vertical part in which the static capacity and the static self-inductance per unit length ($\bar{C}$ and $\bar{L}$) are equally distributed over the whole length. At the same time a self-inductance L_2 and a condenser of capacity C_2 are inserted at the earthed bottom end of the antenna, and at the top we suppose a self-inductance L_1 and capacity C_1 attached in addition. The inductance L_1 is generally not found in actual antennæ, but is inserted here to make the problem symmetrical.

The first things to find are the distributions of the current and of the potential over the vertical part, and also the possible wave lengths with which the whole system can vibrate.*

Taking the vertical as axis of x, with the zero point at the bottom of the vertical wire (*see* Fig. 1), whose length is l; calling the current at any part i and the potential v, we have to find expressions for i and v which will be a solution of the equations,

$$\left.\begin{aligned}\bar{L}\frac{di}{dt}&=-\frac{dv}{dx}\\ \bar{C}\frac{dv}{dt}&=-\frac{di}{dx}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

with the boundary conditions,

$$\left.\begin{aligned}&\text{for } x=o: && v=-\frac{1}{C_2}\int_0^t i\,dt-L_2\frac{di}{dt}\\ &\text{and for } x=l: && v=\frac{1}{C_1}\int_0^t i\,dt+L_1\frac{di}{dt}\end{aligned}\right\}, \quad . \quad . \quad . \quad . \qquad (2)$$

taking the direction of x from the bottom to the top as positive.

In order to simplify the problem, we assume that undamped oscillations exist in the antenna. This assumption is valid, as the distribution of the current is, as a first approximation, independent of the damping.

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* With a loading at the bottom, but none at the top of the aerial the values of the possible wave-lengths are found by L. Cohen, "Electrical World," LXV., 286, 1915; and W. H. Eccles, "Handbook of Wireless Telegr. and Teleph.," p. 121.

Combining the two equations (1) we are led to the well-known formula,

$$LC\frac{d^2i}{dt^2}=\frac{d^2i}{dx^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and a similar equation in v.

If we call the velocity of the disturbance $a=1/\sqrt{CL}$, the angular frequency of the vibration ω, and take a suitable zero-point of time, the solution of (3) for one single frequency may be written in the form,

$$\left.\begin{aligned} i&=T_0\cos\left(\frac{\omega x}{a}-\beta\right).\cos\omega t \\ \text{and}\qquad v&=V_0\sin\left(\frac{\omega x}{a}-\beta\right).\sin\omega t \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where T_0 and $V_0=\sqrt{(L/C)}\ .\ T_0$ are the maximum current and potential amplitudes respectively at an antinode. This antinode eventually may not be found on the antenna itself, but can always be calculated when the actual current distribution over the antenna is known.

The quantities ω and β are to be found by aid of the conditions (2).

Putting $$\theta=\omega\ /a=2\pi l/\lambda, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

the total vertical capacity

$$C=Cl,$$

and the total vertical inductance

$$L=Ll,$$

then applying (2) to (4) the following expressions are easily deduced :—

$$tg\beta=\left(\frac{1}{C_2\omega}-L_2\omega\right)\sqrt{\frac{C}{L}}=\frac{C}{C_2\theta}-\frac{L_2\theta}{L}, \quad . \quad . \quad . \quad (6)$$

and $$tg\left(\frac{\omega'}{a}-\beta\right)=\left(\frac{1}{C_1\omega}-L_1\omega\right)\sqrt{\frac{C}{L}}=\frac{C}{C_1\theta}-\frac{L_1\theta}{L}. \quad . \quad . \quad (7)$$

Now, putting

$$\frac{\omega l}{a}-\beta=\theta-\beta=\alpha, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

we have $$tg\alpha=\frac{C}{C_1\theta}-\frac{L_1\theta}{L}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and we get the following transcendental equation—

$$tg\theta=\frac{tg\alpha+tg\beta}{1-tg\alpha tg\beta}=\frac{\left(\frac{C}{C_1\theta}-\frac{L_1\theta}{L}\right)+\left(\frac{C}{C_2\theta}-\frac{L\ \theta}{L}\right)}{1-\left(\frac{C}{C_1\theta}-\frac{L_1\theta}{L}\right)\left(\frac{C}{C_2\theta}-\frac{L_2\theta}{L}\right)}. \quad . \quad (10)$$

There are an infinite number of solutions θ to this equation, each one of which corresponds to a natural wave-length of the loaded antenna according to (5) and they can be found when the ratios C/C_1, L/L_1, C/C_2 and L/L_2 are known. The values of θ which satisfy (10) are best found by determining the points of intersection of a curve representing the periodic function $tg\theta$, and the function given by the second member of (10)

When using formula (10) it should be remembered that if there is no loading either at the top or bottom of the antenna C_1, L_1 and L_2 are each to be taken as zero, but C_2 must be taken as infinite.

From (4) and (8) it is clear that α and β have the following physical meanings:—The current and potential at the bottom of the antenna have maximum amplitudes respectively of $T_0 \cos \beta$ and $V_0 \sin \beta$, and the maximum current and potential at the top are $T_0 \cos \alpha$ and $V_0 \sin \alpha$.

We can find the values of θ, α and β experimentally by first measuring the wave-length of an antenna when the load at the top in the form of horizontal wires is present, but with no loads at the bottom. The height l being known (5) gives us θ. For no bottom load we have to take, according to (6), $C_2=\infty$, $L_2=0$; which gives $\beta=0$, so that $\alpha=\theta$. In this case we have, therefore,

$$tg\theta=C/C_1\theta,$$

from which C/C_1 can be found. When, now, any capacity and (or) inductance is inserted at the bottom, leaving the top unaltered, and the wave-length λ' is measured again, we have

$$tg\alpha'=C/C_1\theta',$$

from which α' can be calculated, C/C_1 and θ' being known. Again, β' is found by the subtraction $\theta'-\alpha'=\beta'$.

Formula (10) shows us that it is possible, by varying the loads, to have a constant wave-length with different distributions of the potential and current over the antenna, which results in bringing the same sine or cosine function at different altitudes on the antenna.

This is of much importance in connection with the radiation that is sent off the antenna, as will be seen later on.

To find the *total antenna energy* E we have to compute the value of

$$E=\tfrac{1}{2}\bar{L}\int_0^l i^2dx+\tfrac{1}{2}L_2 i^2_{x=0}+\tfrac{1}{2}L_1 i^2_{x=l} \quad . \quad . \quad . \quad . \quad (11)$$

at the time of maximum current, or, what is the same thing, the value of

$$E=\tfrac{1}{2}\bar{C}\int_0^l v^2dx+\tfrac{1}{2}C_2v_2^2+\tfrac{1}{2}C_1v_1^2 \quad . \quad . \quad . \quad . \quad (12)$$

at the time of maximum potential, v_1 and v_2 being the values of the potential at the top and bottom capacities respectively.

Now (11) is equal to—

$$E=\tfrac{1}{4}T_0^2L\left\{1+\frac{1}{2\theta}(\sin 2\alpha+\sin 2\beta)\right\}+\tfrac{1}{2}T_0^2L_1\cos^2\alpha+\tfrac{1}{2}T_0^2L_2\cos^2\beta, \quad . \quad . \quad (13)$$

and from (12) we get

$$E=\tfrac{1}{4}T_0^2L\left\{1-\frac{1}{2\theta}(\sin 2\alpha+\sin 2\beta)\right\}+\frac{1}{2\theta^2}T_0^2\frac{LC}{C_1}\cos^2\alpha$$
$$+\frac{1}{2\theta^2}T_0^2\frac{LC}{C_2}\cos^2\beta, \quad (14)$$

which can be shown to be the same.

Taking the mean of (13) and (14) we get the more symmetrical expression for the total energy of the loaded antenna :

$$E=\tfrac{1}{4}T_0^2L\left[1+\frac{\cos^2\alpha}{\theta}\left(\frac{L_1\theta}{L}+\frac{C}{C_1\theta}\right)+\frac{\cos^2\beta}{\theta}\left(\frac{L_2\theta}{L}+\frac{C}{C_2\theta}\right)\right] \quad . \quad (15)$$

The quotient of the total electromagnetic antenna-energy by half the square of the maximum current at a current antinode may be called the *dynamic self inductance* L' of the complete system. In a similar way the *dynamic capacity* C' of the total system may be defined. It is clear that both L' and C' vary not only with the loads, but also with the harmonic in which the antenna is vibrating. The energy of an earthed vertical rod (plain Marconi aerial) vibrating in its fundamental wave-length ($\lambda=4l$) or any higher harmonic is well known as being $\tfrac{1}{4}T_0^2L$. As all terms in (15) are essentially positive we see that the energy of a loaded antenna, with the same maximum current value at an antinode, and also the dynamic self-inductance L' are always greater than the corresponding values of the same antenna when unloaded.

As we shall see further on, the radiation depends on the current distribution over the vertical part of the antenna. The original energy stored on the aerial can be increased (*see* (15), by increasing the dynamic self-inductance of the antenna. This can be done, *e.g.*, by increasing the self-inductance L_2 at the bottom. The wave-length and the values of the current over the aerial can then be kept constant by decreasing at the same time the capacity of the condenser C_2 at the bottom in such a way that

$$tg\beta = \frac{C}{C_2\theta} - \frac{L_2\theta}{L}$$

is a constant.

By inserting, therefore, a greater inductance at the bottom and at the same time a smaller capacity in the arrangement of Fig. 1, keeping the wave-length constant we are able to diminish the radiating decrement of an antenna with the well-known result of sharper tuning.

The current distribution over the antenna now being known, we can proceed to find the *radiation* that will be thrown off the radiator.

In calculating this radiation we make the following assumptions :—

1. The vertical dimensions of the loading are small in comparison with the total height of the antenna as is actually the case, so that the contribution to the total radiation from the loadings are negligibly small.

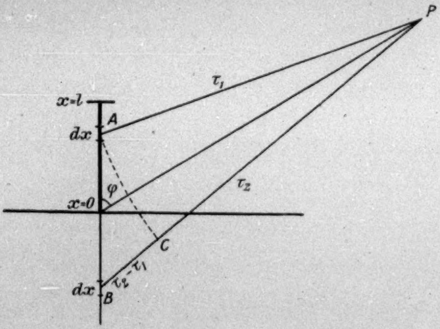

Fig. 2.

2 The antenna is placed on a perfectly conducting earth, with the consequence that the lines of electric force round the radiator may be regarded as ending perpendicularly on the earth.

The electromagnetic field round the antenna may then be found, as shown by Abraham,* by using the electric image of the antenna with respect to the earth *see* Fig. 2, in which AB is the antenna). Taking, now, a point P at a distance from the radiator great in comparison with the wave-length, we can find the intensity of the magnetic and electric fields due to the current in a part dx of the antenna, *and* a part dx of its image. According to Hertz,† the electric force is

$$dE=\frac{2\pi c}{\lambda r_1}i_x dx \sin\varphi \,.\, \sin\left(\omega t-\frac{2\pi r_1}{\lambda}\right)+\frac{2\pi c}{\lambda r_2}i_x dx \sin\varphi \sin\left(\omega t-\frac{2\pi c_2}{\lambda}\right)$$

$$=\frac{4\pi c}{\lambda r}i_x dx \sin\varphi \,.\, \sin\left(\omega-\frac{2\pi r}{\lambda}\right) . \cos\frac{\pi(r_2-r_1)}{\lambda}, \quad (16)$$

where we take $\frac{r_1+r_2}{2}=r=$the distance of P from the bottom of the antenna and where φ is the angle which the radius-vector to P makes with all parts of the antenna. It is clear that the disturbance at P coming from the antenna part dx will arrive there in a different phase from that of the disturbance originating at the same time from the image-part dx.

The magnetic field at P, due to the same sections of the radiator is

$$dH=\frac{4\pi}{\lambda r}i_x dx \sin\varphi \,.\, \sin\left(\omega t-\frac{2\pi r}{\lambda}\right) . \cos\frac{\pi(r_2-r_1)}{\lambda} . \quad . \quad . \quad (17)$$

Now,

$$\frac{\pi(r_2-r_1)}{\lambda}=2\pi\frac{x}{\lambda}\cos\varphi \text{ (\textit{see} Fig. 2, where } r_2-r_1=BC\text{)},$$

so that the electric and magnetic fields at P due to the whole antenna are

$$E=\frac{4\pi T_0}{\lambda r}\sin\varphi \,.\, \sin\left(\omega t-\frac{2\pi r}{\lambda}\right)\int_{x=0}^{x=l}\frac{i_x}{T_0}\cos\left(\frac{2\pi x\cos\varphi}{\lambda}\right)dx, \quad (18)$$

and

$$H=\frac{4\pi T_0}{\lambda r}\sin\varphi \,.\, \sin\left(\omega t-\frac{2\pi r}{\lambda}\right)\int_{x=0}^{x=l}\frac{i_x}{T_0}\cos\left(\frac{2\pi x\cos\varphi}{\lambda}\right)dx. \quad (19)$$

As the electric force is everywhere at right angles to the magnetic force, we find the radiation through a unit surface perpendicular to the radius-vector in a time dt equal to

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* "Physikalische Zeitschrift," II., 329, 1901.

† "Ausbreitung der Electr. Kraft."

$$d\Sigma=\frac{1}{4\pi}EH\,dt=\frac{4\pi cT_0^2\sin^2\varphi}{\lambda^2r^2}\cdot\sin^2\left(\omega t-\frac{2\pi r}{\lambda}\right)\left[\int_0^l\frac{i_x}{T_0}\cos\left(\frac{2\pi x\cos\varphi}{\lambda}\right)dx\right]^2dt. \quad . \quad (20)$$

Now, we found

$$i_x=T_0\cos\left(\frac{\omega x}{a}-\beta\right)\cos\omega',$$

so that the integral in (20) has the value

$$\int_0^l\cos\left(\frac{\omega x}{a}-\beta\right)\cdot\cos\left(\frac{2\pi x\cos\varphi}{\lambda}\right)dx=$$

$$\int_0^l\cos\left(\frac{\theta x}{l}-\beta\right)\cdot\cos\left(\frac{\theta x}{l}\cos\varphi\right)dx=$$

$$\frac{l}{\theta\sin^2\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin(\theta-\beta)-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos(\theta-\beta)\}=$$

$$\frac{l}{\theta\sin^2\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin\alpha-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos\alpha\}.$$

Hence, the mean radiation through the above-mentioned unit surface is

$$d\Sigma=\frac{2\pi cT_0^2}{\lambda^2r^2}\cdot\frac{l^2}{\theta^2\sin^2\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin\alpha-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos\alpha\}^2$$

$$=\frac{cT_0^2}{2\pi r^2\sin^2\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin\alpha-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos\alpha\}^2. \quad . \quad (21)$$

Integrating this expression over a sphere of radius r, and taking half of the result, as radiation exists only over the upper half of the sphere, we get as the expression for the total radiation*,

$$\Sigma=\frac{cT_0^2}{2}\int_0^\pi\frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin\alpha-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos\alpha\}^2d\varphi$$

$$=cT^2\int_0^\pi\frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\cdot\sin\alpha-\cos\varphi\cdot\sin(\theta\cos\varphi)\cdot\cos\alpha\}^2d\varphi, \quad . \quad (22)$$

where T is the R.M.S. current at a current antinode.

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[*] It is clear that the integral given by Abraham in "Phys. Zeitschr.," II., 332, 1901, for the total radiation of a plain rod must be $\int_0^\pi$ instead of $\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$.

Putting R_ρ for the radiation resistance, *i.e.*, the quotient of the total radiation by the *M.S. current in an antinode*, we have

$$\Sigma=T^2R_\rho{}^{*} \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (22\text{A})$$

so that

$$R_\rho=c\int_0^\pi \frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\,.\,\sin\alpha-\cos\varphi\,.\,\sin(\theta\cos\varphi)\,.\,\cos\alpha\}^2 d\varphi.$$

This expression gives the radiation resistance in electromagnetic units. In ohms it is

$$R_\rho=30\int_0^\pi \frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\,.\,\sin\alpha-\cos\varphi\,.\,\sin(\theta\cos\varphi)\,.\,\cos\alpha\}^2 d\varphi. \quad . \quad (23)$$

To evaluate the integral (call it I) in (23), we can proceed as follows

$$I=\int_0^\pi \frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\,.\,\sin\alpha-\cos\varphi\,.\,\sin(\theta\cos\varphi)\,.\,\cos\alpha\}^2 d\varphi$$

$$=\int_{-1}^{+1}\frac{1}{1-u^2}\{\sin\beta+\cos\theta u\,.\,\sin\alpha-u\sin\theta u\,.\,\cos\alpha\}^2 du$$

$$=\tfrac{1}{2}\int_{-1}^{+1}\left(\frac{1}{1+u}+\frac{1}{1-u}\right)\{\sin\beta+\cos\theta u\,.\,\sin\alpha-u\sin\theta u\,.\,\cos\alpha\}^2 du.$$

It is easy to show that the integral with the factor $\frac{1}{1+u}$ is equal to that with $\frac{1}{1-u}$ as factor, so that

$$I=\int_{-1}^{+1}\frac{1}{1+u}\{\sin\beta+\cos\theta u\,.\,\sin\alpha-u\sin\theta u\,.\,\cos\alpha\}^2 du$$

$$=\int_0^2\frac{1}{x}\{\sin\beta+\cos\theta(x-1)\,.\,\sin\alpha-(x-1)\sin\theta(x-1)\,.\,\cos\alpha\}^2 dx \quad . \quad (24)$$

Squaring the function under the integral sign and separating the terms with factors x^{-1}. x^0 and x^{+1}, we can resolve

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[*] The M.S. current *at the bottom* of the antenna in the definition for the radiation resistance is generally used instead of the M.S. current in an antinode. When, however, a condenser of very small capacity is inserted the current at the bottom of the antenna is zero and the radiation resistance becomes infinite, so that the total radiation in that case cannot simply be expressed as a function of the radiation resistance. It is for that reason that the radiation resistance is defined here in relation to the current amplitude in an antinode. The current at the bottom being $T_0\cos\beta$, it is easy to convert the values found here to the more usual expression.

the complete integral I into three separate integrals I_1, I_2 and I_3, of which

$$\begin{aligned}
I_1 &= \sin 2\beta \int_0^2 \frac{\sin \theta x}{x} dx + (1-\cos 2\beta) \int_0^2 \frac{1-\cos \theta x}{x} dx \\
&\qquad -\tfrac{1}{2} \sin 2\beta \int_0^2 \frac{\sin 2\theta x}{x} dx + \tfrac{1}{2} \cos 2\beta \int_0^2 \frac{1-\cos 2\theta x}{x} dx \\
&= \sin 2\beta \int_0^{2\theta} \frac{\sin \varphi}{\varphi} d\varphi + (1-\cos 2\beta) \int_0^{2\theta} \frac{1-\cos \varphi}{\varphi} d\varphi \\
&\qquad -\tfrac{1}{2} \sin 2\beta \int_0^{4\theta} \frac{\sin \varphi}{\varphi} d\varphi + \tfrac{1}{2} \cos 2\beta \int_0^{4\theta} \frac{1-\cos \varphi}{\varphi} d\varphi \\
&= \sin 2\beta \,.\, Si\, 2\theta - \tfrac{1}{2} \sin 2\beta \,.\, Si\, 4\theta + (1-\cos 2\beta) \int_0^{2\theta} \frac{1-\cos \varphi}{\varphi} d\varphi \\
&\qquad + \tfrac{1}{2} \cos 2\beta \int_0^{\theta} \frac{1-\cos \varphi}{\varphi} d\varphi \quad . \quad . \quad (25)
\end{aligned}$$

When using $\theta=\alpha+\beta$, I_2 (with the factor x^0) can be brought into the form,

$$\begin{aligned}
I_2 &= \int_0^2 \Big[-\cos^2 \alpha + \cos \alpha \,.\, \cos \{2\theta(x-1)+\alpha\} - 2 \sin \beta \,.\, \cos \alpha \\
&\qquad \int_0^2 \sin \theta(x-1) \Big] dx \\
&= -\cos^2\alpha \int_0^2 dx + \frac{\cos \alpha}{2\theta} \int_0^{4\theta} \cos (\varphi-\theta-\beta) d\varphi - \frac{2 \sin \beta \,.\, \cos \alpha}{\theta} \int_0^{2\theta} \\
&\qquad \sin (\varphi-\theta) d\varphi \\
&= -2 \cos^2 \alpha + \frac{\cos \alpha}{2\theta} \{\sin (3\theta-\beta) + \sin (\theta+\beta)\} + 0 \\
&= \cos^2 \alpha \left(-2 + \frac{\sin 2\theta}{\theta} \right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)
\end{aligned}$$

I_3 (with the factor x) is

$$I_3 = \cos^2 \alpha \int^2 x \,.\, \sin^2 \theta (x-1) dx = \cos^2 \alpha \left(1 - \frac{\sin 2}{2\theta}\right) . \quad . \quad (27)$$

Therefore, we find as the complete integral—

$$\begin{aligned}
I = I_1 + I_2 + I_3 &= \sin 2\beta \,.\, Si 2\theta - \tfrac{1}{2} \sin 2\beta \,.\, Si\, 4\theta \\
&\quad + (1-\cos 2\beta) \int_0^{2\theta} \frac{1-\cos \varphi}{\varphi} d\varphi + \tfrac{1}{2} \cos 2\beta \int_0^{4\theta} \frac{1-\cos \varphi}{\varphi} d\varphi \\
&\quad + \cos^2 \alpha \left(\frac{\sin 2\theta}{2\theta} - 1 \right) . \quad . \quad . \quad . \quad . \quad (28)
\end{aligned}$$

For small values of θ (*i.e.*, for long wave-lengths) the integrals still occurring in (28) can be expanded in a rapidly converging series as follows :—

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi=\int_0^x \frac{1-1+\frac{\varphi^2}{2'}-\frac{\varphi^4}{4'}+-\ldots}{\varphi}d\varphi=\frac{x^2}{2\cdot 2\,!}+\frac{x^4}{4\cdot 4\,!}-\frac{x^6}{6\cdot 6\,!}+\ldots$$

On the other hand, using the functions integral-cosines we have, when δ is a quantity small in comparison with 1,

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi=\int_0^\delta \frac{1-\cos\varphi}{\varphi}d\varphi+\int_\delta^x \frac{1-\cos\varphi}{\varphi}d\varphi$$

$$=\int_0^\delta \frac{2\sin^2\frac{1}{2}\varphi}{\varphi}d\varphi+\int_\delta^x \frac{d\varphi}{\varphi}-\int_\delta^x \frac{\cos\varphi}{\varphi}d\varphi$$

$$=\tfrac{1}{2}\int_0^\delta \varphi d\varphi+lg_e x-lg_e\delta-\int_\delta^\infty \frac{\cos\varphi}{\varphi}d\varphi+\int_x^\infty \frac{\cos\varphi}{\varphi}d\varphi$$

$$=\tfrac{1}{4}\delta^2-lg_e\delta+lg_e x+Ci\,\delta-Ci\,x.$$

Letting δ approach zero, the limit of this expression becomes

$$\int_0^x \frac{1-\cos\varphi}{\varphi}d\varphi=E+lg_e x-Ci\,x,$$

where E is Euler's constant, and is equal to

$$E=0{\cdot}577216\ldots$$

Applying these results to (28) we can write

$$I=\frac{\sin 2\,\beta}{2}\{2\,Si\,2\theta-Si\,4\theta\}+\frac{\cos 2\,\beta}{2}\{2\,Ci\,2\theta-Ci4\theta-E-lg_e\theta\}$$

$$+E+lg_e2+lg_e\,\theta-Ci\,2\theta+\cos^2 a\left(\frac{\sin 2\theta}{2\theta}-1\right). \quad (29)$$

And we find as the general expression for the total radiation resistance of a loaded antenna, expressed in ohms * :—

$$R_\rho=15\sin 2\beta\,\{2Si\,2\theta-Si\,4\theta\}+15\,\cos 2\beta\,\{2Ci\,2\theta-Ci\,4\theta-lg_e\theta$$

$$-0{\cdot}577216\}+30\left\{\cos^2\alpha\left(\frac{\sin 2\theta}{2\theta}-1\right)-Ci2\theta+lg_e\theta+1.27036\right\} \quad (30)$$

where $\theta=\frac{2\pi l}{\lambda}$ and $\cos\alpha$ and $\cos\beta$ are the ratios of the current amplitudes at the top and bottom respectively of the antenna to the maximum current amplitude at an antinode.

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* Tables of the integral-sine and integral-cosine functions are calculated by J. W. L. Glaisher, and published in the Philosophical "Transactions," London, CLX., 367, 1870.

Formula (30) shows us, as was to be expected, that the radiation-resistance of a loaded antenna and also the radiation from that antenna are dependent not only on the length of the waves which the aerial emits, but also in a high degree on the value of the current at the bottom and top, or, in other words, on how the current at that certain wave-length is distributed over the aerial. It is therefore clear that the radiation from an antenna *cannot* be written in the simple form,

$$\Sigma = AT^2\frac{l^2}{\lambda^2},$$

where A is a constant, and T the R.M.S. current at the base of the antenna, as is usually done in most text-books on wireless telegraphy [*] for it all depends on the manner in which the loading is varied (either at the top or the bottom, or both) to alter the wave-length λ. Even if we assume a flat top antenna —*i.e.*, one with a capacity load at the top in the form of horizontal wires (ship antenna and umbrella antenna)—the total radiation cannot be written as

$$\Sigma = 160\pi^2 T^2 \frac{l^2}{\lambda^2},$$

as is done by Austin, and on which basis he calculated a table of radiation resistances for flat-top antennæ.[†]

For a flat top antenna without any load at the bottom ($tg\beta = 0$) we have from (30),

$$R_\rho = 57{\cdot}0208 + 34{\cdot}5288\, lg_{10}\frac{l}{\lambda} - 15\,Ci\,4\theta + 30\cos^2\theta\left(\frac{\sin 2\theta}{2\theta} - 1\right) \quad (31)$$

It is apparent that the use of the formula—

$$R_\rho = 160\pi^2\frac{l^2}{\lambda^2}$$

to calculate tables of radiation resistances for flat-top antennæ up to ratios $\frac{l}{\lambda} = 0{\cdot}4$, as done by Austin, is an extrapolation of Rüdenberg's results [‡] that is not justifiable, so that the figures given there often differ very much from the values calculated with the aid of our formulæ. Austin does not even state clearly how the variations of λ are to be obtained by varying the top or the bottom loads.

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[*] *See*, *e.g.*, Fleming, "The Principles of Electric Wave Telegraphy and Telephony," 3rd edition, p. 459.

[†] "Journal of the Washington Acad. of Science," Vol. I., No. 7, Nov., 1911. Reprinted in Fleming, "The Principles of Electr. Wave Telegr.," p. 460, and "Jahrbuch der Drahtl. Telegr. u. Teleph.," V., 417, 1912.

[‡] Rüdenberg, "Ann. der Phys.," XXV., 446, 1908.

Rüdenberg's formula is valid only for the case of a very long wave-length emitted from a flat top or umbrella antenna with a capacity at the top large in comparison with the total capacity of the vertical part of the aerial, for then it is possible to deduce the following approximation for the radiation. For no load at the bottom $\beta=0$, and, therefore, $\theta=a$. For a value of λ *large in comparison with the total antenna height* l, θ is, according to (5) a small quantity. Now, the radiation as given by (22A) and (30) is a function of θ only, namely—

$$\Sigma=T^2\left\{15E+30\,lg_e 2+15\,lg_e\theta-15\,Ci\,4\theta+30\cos^2\theta\left(\frac{\sin 2\theta}{2\theta}-1\right)\right\}$$

Expanding the functions of θ occurring in this expression in a series of powers of θ, bearing in mind also that for $\theta<1$:—

$$Ci\,\theta=E+lg_e\theta-\frac{\theta^2}{2\cdot2\,!}+\frac{\theta^4}{4\cdot4\,!}-\frac{\theta^6}{6\cdot6\,!}+\ldots,$$

we find:

$$\Sigma=T^2\{40\,\theta^2-16\,\theta^4+\frac{200}{63}\,\theta^6\ldots\}.$$

Taking only the first term, we have

$$\Sigma=40\,\theta^2\,T^2=160\,\pi^2\,\frac{l^2}{\lambda^2}\,T^2,$$

which is the above-mentioned Rüdenberg's formula and which is only valid for wave-lengths great in comparison with the antenna height.

From (30) and (22A) it follows that *for a constant wave-length* the radiation from an antenna with different loadings at the top and at the bottom, can be expressed in the following way :—

$$\underset{\theta=\text{constant}}{\Sigma}=T^2\{A_1\sin 2\beta+A_2\cos 2\beta+A_3\},$$

where A_1, A_2, and A_3 are constants for a certain ratio of wave-length and antenna height. Bearing in mind that the current distribution is given by

$$i_x=T_0\cos\left(\frac{\omega x}{a}-\beta\right),$$

and taking for x that height on the antenna where a current-node is found, the total energy thrown off the antenna in the form of electromagnetic waves of a constant wave-length, but at different loads, is,

$$\Sigma=T^2\left\{B_1\sin\frac{4\pi x}{\lambda}+B_2\sin\frac{4\pi x}{\lambda}+B_3\right\},$$

where B_1, B_2 and B_3 are constants.

To show the form of this function for two wave-lengths Fig. 3 has been calculated, in which the abscissa gives that part of the height of the antenna at which a current node occurs and the ordinate represents the antenna radiation-resistance at this current distribution, (1) for $\lambda=2l$, and (2) for $\lambda=4l$.

The formula for the radiation resistance of an antenna vibrating in a half wave-length—e g. when x is the height of the current-node, is—

$$\underset{\lambda=2l}{R_\rho}=58{\cdot}28-46{\cdot}32\sin\left(\frac{2\pi x}{l}-1{\cdot}113\right)\Omega,$$

so that the radiation-resistance for this wave-length is a minimum and equal to 11·96 Ω when the current node occurs at a point 42·7 per cent. of the height of the antenna from the base, and the same is a maximum with a value of 104·6 Ω with the node at a height of 92·7 per cent. This example clearly illustrates how the current distribution, even by a constant wave-length, is of importance with regard to the radiation.

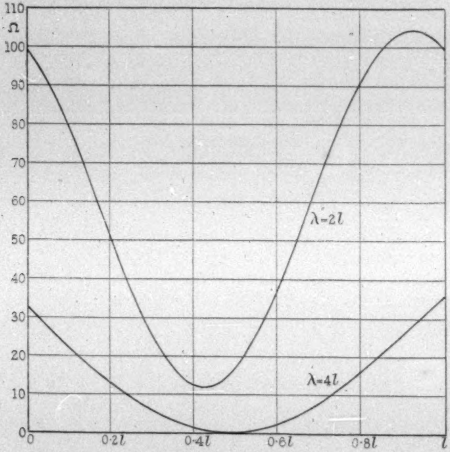

Fig. 3.

In the same Fig. 3 the curve for $\lambda=4l$ shows that there is a different radiation when an antenna vibrates in a quarter wave-length, but

(1) With the current node at the top and antinode at the bottom (plain Marconi aerial), or

(2) Reversed, with the antinode at the top and the node at the bottom, as can be approximated to by putting a large capacity at the top (umbrella antenna) and a very small capacity at the bottom.

In the first instance the radiation resistance is 36·6 Ω,* in the second (according to (30)) only 32·2 Ω.†

It is well known that the usual theory which does not take into account the phase difference at which the waves from various parts of the aerial reach a certain point in space, gives for both the cases just considered the value 40 Ω, a difference of 9·3 per cent. and 21·3 per cent. respectively from the more accurate formula.

It is interesting to find out in what directions the calculated electromagnetic energy is emitted from the antenna. This is best represented graphically. When we write the function under the integral sign in (22) in the form,

$$\frac{1}{\sin\varphi}\{\sin\beta+\cos(\theta\cos\varphi)\,.\,\sin\alpha-\cos\varphi\,.\,\sin(\theta\cos\varphi)\,.\,\cos\alpha\}^2$$

$$=\frac{1}{\sin\varphi}\{f(\varphi)\}^2,$$

polar curves in a plane can be drawn in four different ways having the following peculiarities and formulæ :—

1. If the radius vector r is made proportional to the energy radiated through a unit surface of a sphere of a constant radius described round the sending antenna, the curve has the equation,

$$r=\{f(\varphi)\}^2/\sin^2\varphi \quad . \quad . \quad . \quad . \quad . \quad (32)$$

2. If r is made proportional to the total energy radiated through that zone of a sphere that is comprised between two cones with a slightly different apex-angle, the formula is

$$r=\{f(\varphi)\}^2/\sin\varphi. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

3 r can be made proportional to that distance from the antenna where the radiation through a unit-surface, held at right angles to the radius vector, is a constant, or, in other

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* In accordance with Abraham. *See* Paper cited.

† The usual definition of radiation resistance would give here $R_\rho=\infty$.

words, r can be made proportional to that distance from the antenna where, by a given receiving arrangement, the amount of energy received is a constant We thus have the formula,

$$r=f(\varphi)/\sin\varphi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (34)$$

This surface has the peculiarity that the magnetic and electric forces are constant all over it. This curve can be called the characteristic curve of radiations (analogous to the similar curve in a horizontal plane for directive antennæ such as used by Bellini and Tosi), and represents at the same time the maximum distances in different directions where signals can be received.

4. If $$r=f(\varphi)/\sqrt{\sin\varphi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (35)$$
the area of this surface will be proportional to the total energy radiated.

The accompanying curves are calculated from formula (33), as these curves are apparently the most suitable ones for practical design.

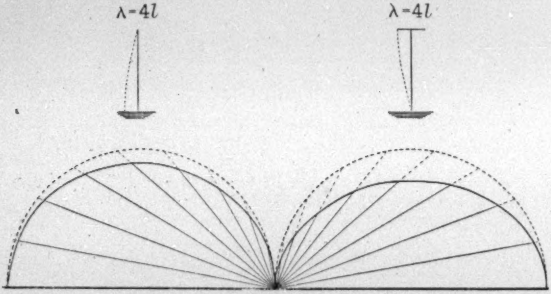

Fig. 4

In Fig. 4 the solid line at the left represents the distances in all directions in a vertical plane from a simple rod-aerial, earthed at the bottom, where the received energy will be constant. The fundamental wave-length is thus equal to $4l$, and the current distribution over the antenna is as indicated at the top of the diagram. As mentioned above, the radiation resistance for this arrangement is 36·6 Ω. At the right side of the same figure is drawn a similar curve for a flat-top antenna, with a very large capacity at the top and a very small capacity at the bottom. The current distribution is then just reversed and the radiation resistance is less than in the former case, and is equal to 32·2 Ω. The dotted lines (for both sides half-circles) represent the curves that would be obtained using the ordinary theory, where the phase differences of waves arriving from different parts of the antenna at a certain point are not taken into account, in which case the excessive value of 40 Ω is found.

It is obvious that when there is a current node in the middle of the antenna the radiation in horizontal directions from one-half of the aerial will, by interference, just annul the radiation from the other half. According to Zenneck's formula,*

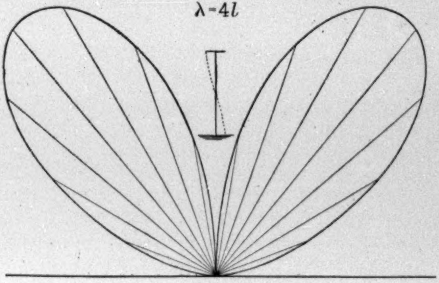

Fig. 5

$$R_\rho = 160\pi^2\left(\frac{a}{\lambda}\right)^2,$$

where

$$a = \frac{\frac{1}{l/2}\int_0^{l/2} i_x d_x}{T_0},$$

and $\frac{l}{2}$=length of the antenna, there would be in this case no radiation at all. In fact, there will be a radiation, occasionally even of a great amount, but in more upward directions.

I have calculated some diagrams for different wave-lengths, but with a current node in the middle of the antenna. Fig. 5 represents this case for $\lambda=4l$, Fig. 6 for $\lambda=2l$, Fig. 7 for $\lambda=4/3l$, Fig. 8 for $\lambda=l$ and Fig. 9 for $\lambda=\frac{2}{3}l$. Each of these cases, of

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* Zenneck, "Lehrb. der Drahtl. Telegr.," 1915.

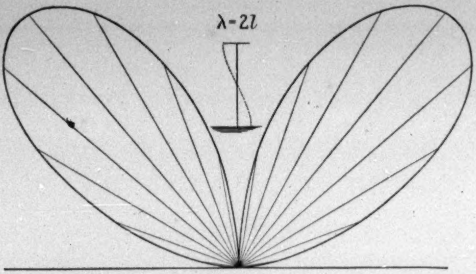

Fig. 6.

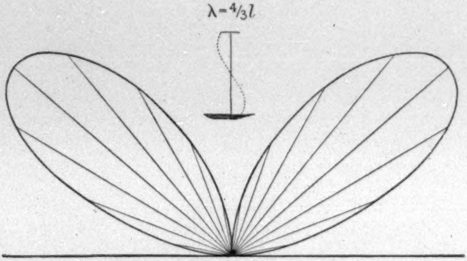

Fig 7.

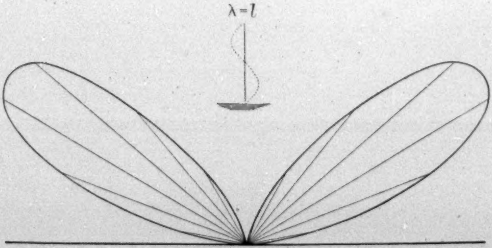

Fig. 8.

course, demands its own peculiar loading at the top and bottom of the antenna. The diagrams show that decreasing the wave-length has the consequence that the radiation will principally be confined more and more between two cones with a smaller difference of apex angle, so that the characteristic curve becomes more favourable, while at the same time the main part of the radiation is directed in a lower direction. Formula (30) shows that the total amount of radiated energy in each case increases with the decrease of the wave-length. Making the wave-length still shorter (by using a higher harmonic) it appears that the radiation will split up in different parts as shown by Fig. 9, in which $\lambda=\frac{2}{3}l$.

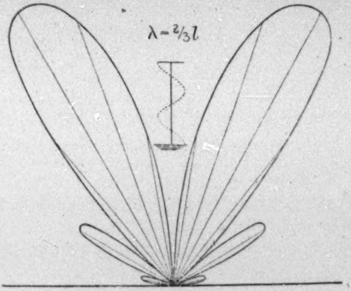

FIG. 9.

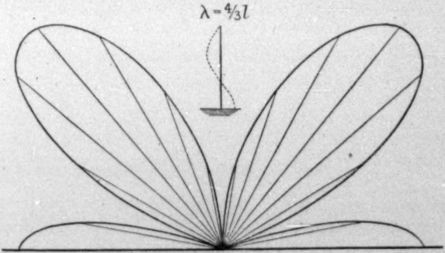

FIG. 10.

Exciting an antenna in a higher harmonic in such a way that there is no current-node at the middle of it, gives a radiation split up again in several parts, of which one will be in the horizontal direction. This is shown in Fig. 10 for a wave-length $\lambda=4/3l$ on a vertical rod without any special loading, with the consequent current distribution over the aerial as indicated in the diagram.

It must be mentioned that all curves of the forms given by (32), (33), (34) and (35) are independent of the distance from the antenna, if only that distance is large in comparison with the dimensions of the antenna and the wave-length used, for which case the Poynting vector is a straight line.

The theoretical possibility of loading an antenna and exciting it in such a way that no radiation will occur in horizontal directions may experimentally be of value to detect the existence of ionised refracting layers in the atmosphere as calculated by Eccles,* for which case the current in a receiving antenna would increase to a certain extent, while reception is made at a greater distance from the sending-station.

Pender Electrical Laboratory,
University College, London.

ABSTRACT.

The Paper consists of a mathematical treatment of the subject, the following being some of the conclusions arrived at: The radiation resistance of a loaded antenna, and also the radiation from the antenna, are dependent not only on the wave-length, but on the current values at the top and bottom. The radiation cannot, therefore, be written $\Sigma=\frac{AT^2l^2}{\lambda^2}$ where A is constant and T is the R.M.S. current at the base, as is done in most textbooks. Rüdenberg's formula for flat-top or umbrella antennæ is valid only for very long wave-lengths, with a capacity at the top of the antenna very large compared with that of the vertical part, and Austin's table of radiation resistances up to ratios of $\frac{l}{\lambda}=0{\cdot}4$ is based on an unjustifiable extrapolation of Rüdenberg's results.

The Paper also treats of the directions in which the energy is most strongly radiated under different conditions.

DISCUSSION.

Prof. J. A. FLEMING communicated the following remarks: The Paper seems to me to be valuable and the analysis correct. The formula for the radiation resistance of a flat-topped antenna, which is commonly called Rüdenberg's formula, has no claim to be attributed to him; it can be at once derived from Hertz's expressions for the forces due to a small oscillator. As I have shown in my book on the " Principles of Electric Wave Telegraphy," third edition, Hertz's reduced expressions are obtained on the conditions that the current has the same value at all points in the rod of the dumb-bell oscillator ; and hence applies also to the reduced formula for radiation resistance. The polar radiation curves which the author has delineated for the higher harmonics of the loaded antenna seem consistent with the delineation of the field of electric force round a rod oscillator given by F. Hack from the equations of Abraham (*see* plate VI., Chapter V., " Principles of Electric Wave Telegraphy "). The possibility of projecting upwards in an inclined direction what is virtually a beam of electric radiation is interesting. By observing where the beam comes to earth it may be possible to determine the height of the reflecting masses of ionised air. There is, however, a certain difficulty in exciting any required high harmonic in an antenna, especially if the capacity and inductance are chiefly located at certain points. When the possibility of experimental work returns it will be interesting to attempt to confirm these results experimentally.

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* THE ELECTRICIAN, September 19, 1913.

A Demonstration of a Method of Preventing Sparking at a rapid Make and Break, which incidentally produces Colloidal Platinum, was given by Dr. A. Griffiths, *at the Meeting on June 8th,* 1917.

The apparatus exhibited was one described by him in the "Philosophical Magazine" for March, 1895, p. 232. The device consists of a series of electrolytic cells placed as a shunt across the spark-gap.

The electrodes consist of platinum and the electrolyte of strong sulphuric acid. The cells polarise, and on making the gap an E.M.F. is introduced opposed to the E.M.F. of the battery, so that the current rapidly diminishes, decomposing the liquid and doing chemical work.

Dr. Griffiths made the following statements:—

1. The platinum cathodes disintegrate and a colloidal solution of platinum is formed.
2. The cathode on the negative side of the spark-gap generally disintegrates to the greatest extent; the next cathode disintegrates less, and so on, the least disintegration occurring in the cell at the positive side of the spark-gap.
3. The cathodes develop, to the naked eye, an appearance as if they were covered with platinum black. Certain plates examined under the microscope seemed covered with numerous craters.
4. The production of gas does not appear to be the same in each of the electrolytic cells in series; sometimes no gas at all appears to be evolved from the most negative cathode.
5. The rate of disintegration of a cathode appears to be small when the cathode is first placed in the sulphuric acid, and appears to increase to a maximum in course of time.
6. One specimen of platinum appears to behave differently from another.

Dr. Griffiths added that by using electrodes made of gold leaf floated on glass with the aid of methylated spirits, he had obtained what was probably colloidal gold. Prof. Hicks, of Sheffield, had told him that he had used electrolytic cells to stop sparking at a pendulum-mercury intermittent contact. He had not observed disintegration of the platinum as far as he (Dr. Griffiths) was aware.

DISCUSSION.

Dr. R. S. **Willows** said he was interested in the chemical aspect of the experiment. It was probable that Dr. Griffiths was the first to obtain colloidal platinum, though he did not call it by that name when the result was originally described. The difference in the intensity of the action in different parts of the circuit reminded him of the similar phenomenon which occurs in a train of X-ray valves, in which most of the action takes place in one valve of the train. Measurements of the potential across each cell would be instructive.

Prof. **Boys** mentioned that he had used electrolytic cells to quench the sparking of a pendulum contact in his gravitation experiments. He had not noticed any action on the platinum. To what did Dr. Griffiths attribute the effect?

Mr. A. **Campbell** communicated the following remarks: Many years ago Helmholtz introduced the use of a condenser, shunted by a suitable resistance, across the spark gap for the purpose of suppressing the spark. More recently a better system was employed by the engineers of the G.P.O., who kindly communicated it to the National Physical Laboratory, where it has been used with the greatest success on all kinds of spark-gaps (from 1 break per second up to 1,000 per second). This consisted in putting the resistance in series with the condenser. By adjusting the value of the resistance to suit the particular circuit it is nearly always possible to suppress the spark. There is no leak across the break, as in the shunt system.

Dr. **Griffiths**, in reply to Prof. Boys, said that he thought the disintegration was due to the inductive rush of the current, but many points required investigation.

XXII. *The Determination of Coma from a Central Ray.* By T. SMITH, *B.A.* (*From the National Physical Laboratory.*)

RECEIVED MAY 1, 1917.

THE optical sine condition, due to Clausius, but more generally associated with the name of Abbe, embodies a result of great importance to the lens designer, since it enables him to form an estimate of the degree to which a system is corrected for coma * without having to calculate any rays other than those necessary to determine the spherical aberration. The extent of the departure from the sine condition is not in itself a measure of the amount of coma. Investigations of the meaning which should be given to a departure from the sine condition when spherical aberration is present have been made by Conrady † and Chalmers.‡ The conclusion reached by Conrady is that departure from the sine condition produces a proportionate zonal change of magnification. That this is true for rays lying in a principal plane is indeed suggested by the well-known law that $\mu h \sin \theta$ is invariant on refraction, where μ is the refractive index, θ the inclination of the ray to the axis, and h a short length perpendicular to the axis at the point where it is met by the ray. Such a statement of the meaning of the condition may suffice to give an indication of the extent to which it is permissible to depart from the condition, but it is evidently not a quite exact statement. For consider a system which is quite free from aberration on the axis, but which does not satisfy the sine condition. If this departure from the sine condition involved only zonal differences of magnification, the image of a point off the axis would be a straight radial line, an effect inconsistent with the essential character of coma, which in this case is the only aberration the departure from the sine condition involves.

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* Coma is to be understood to include all those aberrations, the presence of which influences the fulfilment or non-fulfilment of the sine condition, excepting those which do not disappear when the object point is situated on the axis of the instrument; the latter are referred to as spherical aberrations. Generally speaking, rays from a given object point meeting the stop in an arc of a circular zone will intersect the image plane in a circular arc of equal angular extent when spherical aberration alone is present. When coma alone is present the same rays will meet the image plane in an arc of twice that angular extent.

† "The Optical Sine-Condition," Monthly Notices of R.A.S., Vol. LXV., p. 501.

‡ "The Sine Condition in Relation to the Coma of Optical Systems," Proc. Phy. Soc., Vol. XXII., p. 1.

Chalmers expresses his results in the form :—

Departure from sine condition

$=a\times$ longitudinal spherical aberration

$+b$(1/3 coma of first order $+$1/5 coma of second order

$+$ . . . .),

where a and b are constants.

He assumes that the effect of coma in the image plane may be represented by

$$\frac{\delta x}{x}=i_1(2\ \cos^2\psi+1)\sin^2 A+i_2(4\cos^2\psi+1)\sin^4 A+\ldots,$$

$$\frac{\delta y}{x}=i_1(2\sin\psi\cos\psi)\sin^2 A+i_2(4\sin\psi\cos\psi)\sin^4 A+\ldots,$$

where x is the distance from the axis at which a perfect image would be formed, δx and δy are the displacements in the image plane due to coma of the intersection point of a ray from a zone subtending an angle $2A$ at the axial point of the image plane, and ψ is the angle made by the axial plane through the point in which the ray meets the zone with the axial plane through the object point—*i.e.*, with the plane $y=0$. It will be shown that these assumptions involve a special interpretation of what is to be understood by the term zone.

The problem may be simply investigated by the aid of a potential function. Let the origins of co-ordinates in the object and image spaces be the point in which the axis meets the object plane, and its image formed by paraxial rays respectively. The co-ordinates and corresponding direction cosines in the object space will be denoted by x, y, z and l, m, n, and in the image space by the same letters accented. The axes of x and x' coincide with the axis of symmetry of the system. The length of the path between a given point of the object plane and the foot of the perpendicular to the refracted ray from the origin in the image space being denoted by U,* the potential function to be used will be B, where $U+B=$constant The variables being y, z, m', n', it is not difficult to show that the direction cosines of the incident ray and the point of intersection of the refracted ray with the paraxial image plane are given by

$$m=\frac{\partial B}{\partial y},\ n=\frac{\partial B}{\partial z},\ y'=\frac{\partial B}{\partial m'},\ z'=\frac{\partial B}{\partial n'},$$

---

* U is called by Bromwich the "modified characteristic function."

where the initial and final refractive indices are taken to be unity. The extension to the more general case will be easily obtained later.

Since the system is symmetrical about the x axis B must evidently be a function of y^2+z^2, $m'y+n'z$, and $m'^2+n'^2$ only. Taking the focal length to be unity, B may be written in the form—

$$\begin{aligned}
B=g(m'y+n'z)&+\tfrac{1}{2}g(y^2+z^2)\\
&+\tfrac{1}{4}\alpha_1(m'^2+n'^2)^2+\tfrac{1}{6}\alpha_2(m'^2+n'^2)^3+\tfrac{1}{8}\alpha_3(m'^2+n'^2)^4+\;..\\
&+(m'y+n'z)\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2\\
&\qquad\qquad+\beta_3(m'^2+n'^2)^3+\ldots\}\\
&+(y^2+z^2)\{\gamma_1(m'^2+n'^2)+\gamma_2(m'^2+n'^2)^2+\gamma_3(m'^2+n'^2)^3+\,..\}\\
&+(m'y+n'z)^2\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+\,..\}\\
&+\text{ terms involving higher powers of } y \text{ and } z.
\end{aligned}$$

The magnification of the object produced by paraxial rays is evidently g. The image point for the ray defined by (y 0, m', n') is—

$$\begin{aligned}
y'=gy&+m'\{\alpha_1(m'^2+n'^2)+\alpha_2(m'^2+n'^2)^2+\alpha_3(m'^2+n'^2)^3+\,..\}\\
&+y\{\beta_1(3m'^2+n'^2)+\beta_2(5m'^2+n'^2)(m'^2+n'^2)\\
&\qquad\qquad+\beta_3(7m'^2+n'^2)(m'^2+n'^2)^2+\,..\}\\
&+\text{ terms involving higher powers of } y.
\end{aligned}$$

$$\begin{aligned}
z'=n'&\{\alpha_1(m'^2+n'^2)+\alpha_2(m'^2+n'^2)^2+\alpha_3(m'^2+n'^2)^3+\,..\}\\
&+2m'n'y\{\beta_1+2\beta_2(m'^2+n'^2)+3\beta_3(m'^2+n'^2)^2+\,..\}\\
&+\text{ terms involving higher powers of } y.
\end{aligned}$$

The coma terms in the expressions for y' and z' are those involving y to the first power as a factor when m' and n' assume the values they bear for a ray from the axial point of the object. Let the direction cosines for such a ray be denoted by capital letters L, M, N, L', M', N'. Although m' and n' may differ very little from M' and N', it is not permissible to replace the small letters by capitals in the equations for y' and z', for m' and n' are only actually equal to M' and N' when $y=0$. The exact expressions will be—

$$\left.\begin{aligned}
m'&=M'+yp+y^2p'+\,..\\
n'&=N'+yq+y^2q'+\,..
\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (1)$$

where p, q, p', q', . . . are functions of M' and N'. In the present case only p and q are involved in the determination of coma. The substitution of $M'+yp$ and $N'+yq$ for m' and n' leads to

$$\begin{aligned}
y' &= gy+M'\{\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2 \\
&\qquad +\alpha_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y[(\beta_1+\alpha_1 p)(3M'^2+N'^2)+(\beta_2+\alpha_2 p)(5M'^2+N'^2)(M'^2+N'^2) \\
&\qquad +(\beta_3+\alpha_3 p)(7M'^2+N'^2)(M'^2+N'^2)^2+\ldots \\
&\quad +2qM'N'\{\alpha_1+2\alpha_2(M'^2+N'^2)+3\alpha_3(M'^2+N'^2)^2+\ldots\}] \\
&\quad + \text{terms involving } y^2, \text{ \&c.,}
\end{aligned}$$

and

$$\begin{aligned}
z' &= N'\{\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2+\alpha_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y[2M'N'\{\beta_1+\alpha_1 p+2(\beta_2+\alpha_2 p)(M'^2+N'^2) \\
&\qquad +3(\beta_3+\alpha_3 p)(M'^2+N'^2)^2+\ldots\} \\
&\quad +q\{\alpha_1(M'^2+3N'^2)+\alpha_2(M'^2+5N'^2)(M'^2+N'^2) \\
&\qquad +\alpha_3(M'^2+7N'^2)(M'^2+N'^2)^2+\ldots\}] \\
&\quad + \text{terms involving } y^2, \text{ \&c.}
\end{aligned}$$

Comparison with Chalmers' initial equations shows that he has assumed $p=\text{constant}$ and $q=0$.

The direction cosines of the incident ray are

$$\begin{aligned}
m &= gm'+gy+m'\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2 \\
&\qquad +\beta_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2y\{\gamma_1(m'^2+n'^2)+\gamma_2(m'^2+n'^2)^2+\gamma_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2m'^2y\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+\ldots\}+\ldots
\end{aligned}$$

and

$$\begin{aligned}
n &= gn'+n'\{\beta_1(m'^2+n'^2)+\beta_2(m'^2+n'^2)^2+\beta_3(m'^2+n'^2)^3+\ldots\} \\
&\quad +2m'n'y\{\delta_1+\delta_2(m'^2+n'^2)+\delta_3(m'^2+n'^2)^2+\ldots\}+\ldots
\end{aligned}$$

or, substituting for m' and n' in terms of M' and N'—

$$\begin{aligned}
m &= gM'+gy(1+p)+M'\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2 \\
&\qquad +\beta_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +y\{(2\gamma_1+\beta_1 p)(M'^2+N'^2)+(2\gamma_2+\beta_2 p)(M'^2+N'^2)^2+\ldots\} \\
&\quad +2M'^2y\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad +2M'y(M'p+N'q)\{\beta_1+2\beta_2(M'^2+N'^2) \\
&\qquad +3\beta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}$$

and

$$\begin{aligned}
n &= gN'+gqy+N'\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2 \\
&\qquad +\beta_3(M'^2+N'^2)^3+\ldots\} \\
&\quad +2M'N'y\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad +qy\{\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2+\ldots\} \\
&\quad +2N'(M'p+N'q)y\{\beta_1+2\beta_2(M'^2+N'^2) \\
&\qquad +3\beta_3(M'^2+N'^2)^2+\ldots\} \\
&\quad + \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot
\end{aligned}$$

PHYSICAL SOCIETY OF LONDON.

Proceedings at the Meeting held on Friday, June 22, 1917, at the Imperial College of Science.

Mr. W. R. Cooper, M.A., Vice-President, in the Chair.

A Paper entitled "The Determination of Coma from a Central Ray," was read by Mr. T. Smith.

The absence of coma from an optical system free from spherical aberration can be established by means of the sine condition, from the refraction of rays which originate at the centre of the object. In actual systems spherical aberration is always present to some extent, and the conditions which have to be secured in practical cases do not involve mathematical freedom from spherical aberration and coma, but rather their confinement within predetermined limits. Previous investigations on this subject have been published by Conrady and by Chalmers; the present Paper shows that the conclusions of both require some qualification. Conrady's conclusion, that the sine ratio for any zone gives the exact magnification, holds only for rays in a secondary plane, and one of the further conditions (*a*) that the zone is free from spherical aberration, or (*b*) that the stop is at the principal focus for rays travelling in the negative direction, must also be satisfied. Under these conditions the magnification for rays in a primary plane is $\left(1+\tan\theta'\frac{d}{d\theta'}\right)$ times that for the secondary rays, where $2\theta'$ is the angle subtended by the zone at the centre of the image. The conditions assumed by Chalmers involve the satisfaction of a relation between various quantities, including the curvature and astigmatism of the system; his results may in consequence not apply strictly where very large apertures are involved. Analysis of the general problem shows that the primary and secondary comatic displacements can be accurately derived from the properties of thin oblique pencils of rays. If a central incident ray inclined at θ to the axis is met by an oblique ray in the same axial plane at a distance r from its point of origin, and is refracted at an angle θ' with the axis and meets the image plane at a distance r' from the image of its intersection with the oblique ray, the primary comatic displacement of the oblique ray plus the aberrationless first order displacement is proportional to $r' \sec\theta'/sr \sec\theta$, where $\mu sk/\mu'$ is the magnification in a primary plane for a small normal object at the point of intersection of the two rays. The corresponding secondary displacement is $\rho'/\sigma\rho$, where the Greek letters have the same meaning for secondary rays as the corresponding Roman letters for primary rays. These two quantities determine the coma exactly whatever the spherical aberration may be. It follows that coma is dependent upon the principal surfaces of a lens. When the object is at infinity the coma is determined by the locus of the point of intersection of a ray, incident parallel to the axis, with its refracted portion.

The results established in the Paper form one of a series of reciprocal relations which exist, between the aberrations of an object and those of the effective stop.

DISCUSSION.

Mr. S. D. **Chalmers** pointed out the exact relation which he had given in the Paper quoted by Mr. Smith, and said that an assumption was necessary to apply this to the evaluation of coma. The assumption was that the two rays from an axial point and a neighbouring point to the secondary point on the stop would intersect in the image space substantially on the image of the stop. He thought this assumption was justified, though in exceptional cases it might require careful examination. Apart from such special cases he thought it was perfectly justifiable to use the expansion for the coma in powers of sin A, and to deduce the values in the primary plane from these in the secondary plane. It would be necessary to determine the values for sufficient apertures in the secondary plane to find each of the terms in the coma expansion.

Prof. F. J. **Cheshire** said he was pleased to see that the subject of optical designing was occupying an important place in the work of the Physical Society. The maker who used to turn out the best microscope objectives in Germany would sit with his microscope beside him and a large number of component lenses from which he would try one combination after another until a satisfactory result was reached. In this way some excellent objectives and a large number of mediocrities were obtained. Abbe set out to kill this trial and error method, and introduced at the Zeiss works the system of determining mathematically the complete structural data beforehand. In this country a trial and error method was still largely employed. An approximate design was computed and a trial instrument was made. This turned out badly, and the designer had another try, and so on. First-class results were impossible on these lines. It was true that we had two or three first-class designers in the country whose designs could safely be made up without trial; and this must ultimately be aimed at by every optician if we are to take a prominent position in optical work.

Mr. **Blakesley** said that an idea was involved in Mr. Cheshire's remarks which he had long held—viz., that a lens should only be used to form an image in the one particular position for which it is designed.

Mr. T. **Smith**, in reply to Mr. Chalmers, said that the exact relation mentioned led at once to an expression equivalent to the alternative formula (17). If expressed in this form the equation was exact. The transformation given in Chalmers' Paper, which involved the neglect of a term, was apparently made to enable the primary coma to be found on the supposition that expansion in terms of sin A would separate out the coma of different ratios. Such a method could be applied provided the spherical aberration were also analysed into terms of different orders in sin A, but not otherwise. The process would then be a comparatively laborious method of reaching the result given immediately by either of the formulæ for the primary comatic displacement contained in the present Paper. The formula, given by Chalmers was correct in the absence of spherical aberration ; if the spherical aberration of the first n orders were negligible, no serious errors in the estimation of the first $(n+1)$ orders of coma from his formula would arise, but appreciable spherical aberration of the nth order would throw out the estimate of coma of the $(n+1)$th and subsequent orders.

A Paper, entitled " Chromatic Parallax and its Influence on Optical Measurements," was read by Mr. J. GUILD.

The Paper describes various parallax effects observable when objects illuminated by light of different colours are seen through a "pinhole" of smaller diameter than the pupil of the eye. The effects are due to the well-known chromatic aberration of the eye ; and it is shown that in most optical instruments of precision the necessary conditions for parallax of this nature are present when observations are made in the blue and violet ; and that this results in a serious diminution of the accuracy of such measurements. Several methods of obviating the difficulty are described.

DISCUSSION.

Prof. CHESHIRE said it was possible that the author had laid his finger on one of the main causes of the discrepancies which were found in many important determinations, those of refraction and dispersion of optical glass for example, between the results of different experimenters. He was struck by the expedient of using a fibre or wire at the slit, and was surprised that this method had not been employed before. Some interesting stereoscopic effects ought to be observed if two differently coloured objects were viewed through pinholes in front of each eye. If the pinholes were further apart than the centres of the pupils, the distance between the blue image in one eye and that in the other would be less than the distance between the red ones, and the blue object would appear further off. If the pinholes were closer than the centres of the pupils, the reverse would be the case, and the blue object would appear to be closer. In many opticians' windows the word FRIEND was exhibited, the letters F I and N being blue, while R, E and D were coloured red. On looking at this the red letters appeared to stand in front of the others. Prof. Von Rohr had once told him this was a true stereoscopic effect, the images on the two retinæ of a blue letter being closer together than the images of a red letter on account of the aberrations of the eye. He had not given much thought to it himself, and would be interested if the author could explain the effect.

Mr. T. H. BLAKESLEY said that if one formed a spectrum of a white light by a prism with refracting edge perpendicular to the slit, a drawn out image of the latter was obtained, which was white at the middle and coloured at the extremities. If this were now viewed through a pinhole which was moved to one side, he supposed the blue and red extremities would be deflected in opposite directions, while the central portion would be drawn out into a continuous spectrum. He did not think the refractive properties of the eye were sufficiently well known to justify actual numerical calculations of the Gaussian system.

Mr. S. D. CHALMERS believed there had been a number of accurate determinations of the chromatic aberration of the eye. He was in the habit of assuming an aberration of ¾ Diopter (from C to F) in designing eyepieces, and for other purposes. He was interested in the author's wide slit method. He had himself found it better than using a fine slit to employ one of appreciable width and bisect it with the crosslines. Why not use a thick vertical wire in the eyepiece and a slit of three times the width. The setting obtained by bisecting the broad line with the wire was a very accurate one.

Mr. T. Smith pointed out that the result of § 5—viz., that the total parallax obtainable with a small fixed stop in front of the eye was independent of the positions of the objects—could be shown very simply. Owing to the stop the mean ray from any object was refracted at the same point of the refracting surface (approximately). The deviation of a blue ray incident at this point is independent of the angle of incidence, and $=\beta h$, where h is the distance from the axis, and $\beta=(\mu_b-1)/\mu_b r$, r being the radius of the surface. Similarly, the deviation of a red ray incident at the same point is ρh, where $\rho=(\mu_r-1)/\mu_r r$. The difference between the deviations of the two rays $=(\beta-\rho)h$, and is the parallax when the stop is at a distance h from the axis. Since β and ρ are independent of the inclinations of the incident rays, the parallax is independent of the distances of the objects from the eye. Mr. Guild's work in connection with these phenomena and their elimination in optical measurements had had important results. The precautions which he suggested had now been in operation at the laboratory for a considerable time, and the departmental records showed that the consistency of certain important determinations had been increased by at least five times. It was obviously essential to remove chromatic parallax if only because the errors due to it were so unsystematic.

Lieut.-Col. J. W. Gifford (communicated remarks) asked if the author had worked with an ordinary Ramsden eyepiece and shadow pointer—that is, a knife-edge pointer reversed so that there is no reflected light from the pointer. With shutters added so as to block out everything but the line under measurement he had found such an arrangement more accurate than any other he had used. Of course, the projected shadow of the point must be accurately focussed by the eyepiece for each line to be measured.

The Author replied to the points raised in the discussion. For reasons of space the reply is held over to the "Proceedings."

A Paper, entitled "A Note on the Graphs of Bessel Functions of Integral Order," by Mr. B. Hague, was taken as read, in the absence of the author.

A demonstration of the Measurement of Differences of Curvature by Means of Monochromatic Interference Rings, and the use of Steel Spheres as Reference Standards of Curvature, was given by Mr. S. D. Chalmers.

He also showed a new method of measuring the central separation of two curved surfaces. The interference ring pattern changes with the angle of incidence, and by observing the number of rings which disappear into the centre as the angle of incidence increases, the separation is determined. Both experiments can be arranged in a simple form as laboratory teaching experiments.

It will be convenient to adopt a more concise notation. Put

$$a=\alpha_1(M'^2+N'^2)+\alpha_2(M'^2+N'^2)^2+\alpha_3(M'^2+N'^2)^3+\,..$$
$$a'=2\{\alpha_1+2\alpha_2(M'^2+N'^2)+3\alpha_3(M'^2+N'^2)^2+\,..\}$$
$$b=g+\beta_1(M'^2+N'^2)+\beta_2(M'^2+N'^2)^2+\beta_3(M'^2+N'^2)^3+\,..$$
$$b'=2\{\beta_1+2\beta_2(M'^2+N'^2)+3\beta_3(M'^2+N'^2)^2+\,..\}$$
$$c=g+2\{\gamma_1(M'^2+N'^2)+\gamma_2(M'^2+N'^2)^2+\gamma_3(M'^2+N'^2)^3+\,..\}$$
$$d=2\{\delta_1+\delta_2(M'^2+N'^2)+\delta_3(M'^2+N'^2)^2+\,..\}$$

so that

$$\left.\begin{aligned} y'&=aM'+y\{ap+b+M'(M'p+N'q)a'+M'^2b'\}+\,..\\ z'&=aN'+y\{aq+N'(M'p+N'q)a'+M'N'b'\}+\,..\end{aligned}\right] \quad .. \quad (2)$$

$$\left.\begin{aligned} m&=bM'+y\{bp+M'(M'p+N'q)b'+c+M'^2d\}+\,..\\ n&=bN'+y\{bq+N'(M'p+N'q)b'+M'N'd\}+\,..\end{aligned}\right] \quad . \quad (3)$$

The quantities corresponding to y', z', m, n for the central ray are denoted by Y', Z', M, N. They are obtained by putting $y=0$ into equations (2) and (3). The quantity a defines the central spherical aberration and b is the sine ratio, which is constant when the sine condition is satisfied.

Equations (1), (2) and (3) show that p and q serve to define with which ray from $(y, 0)$ the central ray is compared. Suppose the two rays intersect in the object space at a point distant ρ from the origin. The conditions

$$\frac{l}{L}=\frac{m}{M-y/\rho}=\frac{n}{N}$$

are then satisfied.

The relation $\frac{l}{L}=\frac{n}{N}$ gives

$$\{p(b^3+b')+b^2c+d\}M'N'+q\{b-b^3M'^2+b'N'^2\}=0, \quad . \quad (4)$$

and $\frac{m}{M-y/\rho}=\frac{n}{N}$ reduces to

$$\frac{1}{\rho}+c=b\left(\frac{M'}{N'}q-p\right). \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Similarly, if the emergent rays meet in a point distant ρ' from the intersection of the central ray with the paraxial image plane,

$$\frac{\rho'L'}{l'}=\frac{Y'-y'+\rho'M'}{m}=\frac{Z'-z'+\rho'N'}{n'}$$

The equality of the last two expressions yields

$$(\rho'+a)\left(\frac{M'}{N'}q-p\right)=b, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and the other condition leads to

$$\left\{\frac{\rho'}{1-M'^2-N'^2}+a+(M'^2+N'^2)a'\right\}(M'p+N'q)+M'b+M'(M'^2+N'^2)b'=0. \quad . \quad (7)$$

If both incident and emergent rays meet, equations (5) and (6) give

$$\left(\frac{1}{\rho}+c\right)(\rho'+a)=b^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

as the equation for finding the point of intersection of the refracted rays when that for the incident rays is known. In order that the rays should actually intersect in this way p and q must receive values which enable both the conditions (4) and (7) to be satisfied. In general, whether the rays meet or not (8) is the relation determining the position of the radial focal line for a given object point.* This result does not hold for rays lying in a primary or axial plane, for which case q should be eliminated between (4) and (5) and N' put equal to zero in (7). The corresponding equations are, therefore,

$$\frac{1-b^2M'^2}{r}+p(b+M'^2b')+c+M'^2d=0, \quad . \quad . \quad . \quad (9)$$

$$\frac{r'p}{1-M'^2}+p(a+M'^2a')+b+M'^2b'=0, \quad . \quad . \quad . \quad (10)$$

or, eliminating p,

$$\left(\frac{1-b^2M'^2}{r}+c+M'^2d\right)\left(\frac{r'}{1-M'^2}+a+M'^2a'\right)=(b+M'^2b')^2, \quad (11)$$

where r and r' have been written instead of ρ, ρ' to distinguish the different cases. Equation (11) fixes the position in which the central ray is met by the transverse focal line for a given object point.

Since from equation (4) q involves $M'N'$ as a factor, the value of q is seen from equations (2) to be quite immaterial as regards the refraction of rays associated with the ray $M'=0$. The points, therefore, of the comatic ring lying in the plane of the object point are determined by p alone. The value of q, however, does affect the shape of the ring. Equation (4) shows that the ring from a zone cannot be assumed to be circular unless the zone is so defined that $q=0$. Moreover, under these conditions p can only be a constant if certain relations hold among the coefficients of the system, and these coefficients evidently involve the curvature and astigmatism in the system. It is, therefore, not permissible to assume that coma can be divided into terms of various orders which will automatically be resolved into components having displacements for rays in primary and secondary planes in the ratios 3 : 1, 5 : 1, 7 : 1, &c., by expanding in powers of the sine of the semi-angular aperture of a symmetrically placed circular zone. Chalmers' formula, though probably reliable in most cases, will consequently not be exact for large apertures. Assuming, however, a definition of a zone which leads to $q=0$, it is possible to find the two image points in the primary or axial plane, and so determine exactly the amount of the coma present.

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* This equation may be compared with equations (9) of p. 508, Proc. Phys. Soc., Vol. XXVII.

Take first the ray in the secondary plane $M'=0$, for which

$$y'=y(ap+b),\ z'-Z'=0.$$

From equation (6)

$$ap+b=-p\rho' . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12)$$

The distances from the axis of the conjugate points on the ray are, for the object space,

$$\rho N=\rho bN',$$

and for the image space

$$(\rho'+a)N'.$$

Let the ratio of these heights be σ ; then

$$\rho'+a=\rho b\sigma,$$

and by comparison with (12)

$$p\rho\sigma=-1,$$

so that

$$\frac{y'}{y}=ap+b=\frac{\rho'}{\rho\sigma} . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13)$$

Now, consider rays in the primary plane for which $N'=0$. Equations (2) give

$$z'=0,$$

and

$$\begin{aligned} y'-Y' &= y\{ap+b+M'^2(a'p+b')\} \\ &= -y\frac{r'p}{1-M'^2}, \end{aligned}$$

by equation (10).

The p in this instance is as a rule not the p of the previous case. From equations (3) and (9)

$$\begin{aligned} m-M &= m-bM' \\ &= y\{bp+M'^2b'p+c+M'^2d\} \\ &= -y\,.\,\frac{1-M^2}{r}. \end{aligned}$$

Also from (1)

$$m' - M' = yp,$$

and, therefore,

$$\frac{m-M}{m'-M'} = -\frac{1-M^2}{p'},$$

and

$$\frac{y'-Y'}{y} = \frac{r'(1-M^2)}{r(1-M'^2)} \cdot \frac{m'-M'}{m-M},$$

$$= \frac{r'}{rs}\sqrt{\frac{1-M^2}{1-M'^2}}, \quad . \quad . \quad . \quad . \quad . \qquad 14)$$

where s is the reciprocal of the angular magnification at the conjugate points determined by r and r'. If $\sin\theta$ and $\sin\theta'$ are written for M and M' respectively the conditions for freedom from coma may evidently be written,

$$\frac{\rho'}{\sigma\rho} = \frac{r' \sec\theta'}{sr \sec\theta} = \text{constant for all rays} \quad . \quad . \qquad (15)$$

These conditions hold whatever the spherical aberration on the axis may be.

The quantities

$$\frac{r' \sec\theta'}{sr \sec\theta} \text{ and } \frac{\rho'}{\sigma\rho}$$

may evidently be looked upon as the magnifications in the primary and secondary directions for small pencils, of which the ray (M', N') is a member If these are greater than the magnification for paraxial rays the distance of the comatic ring from the axis will be greater than that of the image point for a ray through the centre of the objective. It is possible for one of these magnifications to be greater and the other less than the paraxial magnification, a case which shows the importance of finding both quantities if a correct estimate of the magnitude of the comatic patch is to be obtained.

If the external refractive indices differ it is necessary to regard g, s and σ as the reciprocals of angular magnifications. The quantities which have hitherto been called lengths will then be lengths multiplied by refractive indices As, however, only the ratios of $\frac{y'-Y'}{y}$, $\frac{r'}{r}$, $\frac{\rho'}{\rho}$ to one another are involved in the final result, and these are all altered in the same proportion, it will be permissible to look upon the quantities that have been called lengths as absolute lengths, making g a linear magnification, while s and σ remain as before, reciprocals of angular magnifications, or linear magnifications multiplied by μ'/μ, where μ and μ' are the refractive indices of the object and image spaces respectively.

The final conditions in their general form might well have been suggested by the theory of oblique pencils of rays. Let Fig. 1 represent an axial plane of the system. OO' is the axis, and OP, OQ incident rays arising from the axial point O, which are refracted as $P'O'$, $Q'O'$. Other rays, YP and YQ, start from Y, a point in the plane of the diagram on the perpendicular to the axis through O, and after refraction emerge as $P'Y'$ and $Q'Y'$. If the distance OY is small, and P and Q are equidistant from the axis, and in the same normal plane, P' and Q' will lie on another normal plane, and be equidistant from the axis, and Y' will lie on the perpendicular to the axis through O'. P, P' and Q, Q' are pairs of conjugate points, and provided the angles are small the ratio of the angle between the two incident rays OP, YP to the angle between the corresponding emergent rays $P'O'$, $P'Y'$ is constant. Call this ratio s, so that $\widehat{OPY}=s\,.\,\widehat{O'P'Y'}$. Let ON, $O'N'$ be perpendiculars on to the rays PY, $P'Y'$. Then $ON=OP\ \widehat{OPY}$ and $O'N'=P'O'\,.\,\widehat{O'P'Y'}$, or

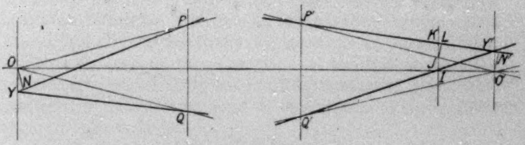

FIG. 1.

$$\frac{ON}{O'N'}=\frac{s\,.\,OP}{P'O'}.$$

Now, if the angles between OP and $P'O'$ and the axis are θ, θ', the angles $\widehat{YON}$ and $\widehat{Y'O'N'}$ are also θ, θ'. Thus, $ON=OY\cos\theta$, $O'N'=O'Y'\cos\theta'$, or

$$\frac{O'Y'}{OY}=\frac{P'O'\sec\theta'}{s\,.\,OP\sec\theta}.$$

If, instead of considering the length $O'Y'$ in the plane in which the central ray meets the axis another normal plane, such as that meeting the axis in I were taken, the result obtained would be—

$$\frac{JK}{OY}=\frac{P'J\sec\theta'}{s\,.\,OP\sec\theta}.$$

Figs. 2 and 3 relate to rays in a secondary plane. In Fig. 2 the rays from O and Y are seen superposed. Fig. 3 is a view in which the planes containing both central and oblique secondary rays appear as straight lines. The secondary conjugate foci on the central rays are p, p' and q q'. The point of union of the two oblique rays is H on the line from O' through Y' (Fig. 1). The ratio of the angles $\widehat{OpY}$, $\widehat{O'p'H}$ is a constant σ so long as these angles are small. Thus,

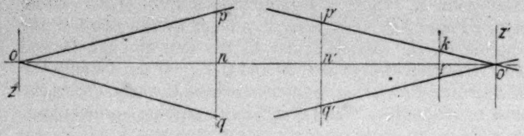

FIG. 2.

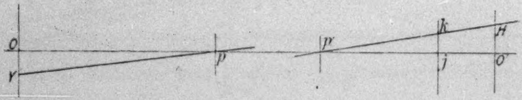

FIG. 3.

$$\frac{HO'}{OY}=\frac{p'O'\,.\,\widehat{O'p'H}}{Op\,.\,\widehat{OpY}}=\frac{p'O'}{\sigma\,.\,Op}$$

or, if measurements are to be made in the plane through I and the rays $p'O'$ and $p'H$ meet this plane in j and k respectively,

$$\frac{jk}{OY}=\frac{p'j}{\sigma\,.\,Op}$$

Now, if h is a short length on a straight line through p normal to the plane Opn, and h' is its image, and therefore a length on a straight line through p' normal to the same plane,

$$\sigma=\frac{\mu'h'}{\mu h}.$$

But the lengths h' and h can be determined by rotating the primary plane containing p through a small angle about the axis, that is to say,

$$\sigma=\frac{\mu'h'}{\mu h}=\frac{\mu'\,.\,p'n'}{\mu\,.\,pn}.$$

The length $O'Y'$ gives the distance from the axis at which the rays YP, YQ in a primary plane unite, and $O'H$ gives the distance at which the rays Yp, Yq in a secondary plane unite. When coma is present but there is no spherical aberration, as in the plane through O', the difference between $O'Y'$ and $O'H$ gives the diameter of the comatic circle. The conditions for freedom from coma are the equality of $O'Y'$ and $O'H$ to one another and to the distance from O' of the paraxial image of Y. When spherical aberration is present, as when the measurements are made in the plane through I, the light from Y through the zone $PpQq$ will meet the image plane in a curve exactly similar to that associated with the rays from O through this zone if JK and jk are equal. The condition of freedom from coma throughout the lens aperture is that for every zone JK and jk shall be equal to the displacement for paraxial rays. The analysis based on the potential function B shows that the conditions to which this geometrical treatment leads hold rigorously no matter how great the inclination of the incident beams of light may be.

In the application of the results to the determination of the amount of coma present in a system, the ray (M', N') will usually pass near the margin of the aperture. The limiting stop may be a component lens or a hole in an opaque screen placed in the object space, the image space, or in an intermediate position between some of the component lenses. One of the intersection points will usually be selected in the plane of the stop or an image position corresponding approximately to that plane. It then only remains to find the two conjugate points and the corresponding primary and secondary magnifications to determine the coma. The determination of these conjugate points and magnifications is a type of calculation with which optical computers are quite familiar, and so offers no difficulty.

The effective stop of many important systems is in the neighbourhood of the principal points. The properties usually ascribed to the principal planes will not hold for any but paraxial rays, neither in general can the planes be replaced by a pair of surfaces which stand to one another in the relation of object and image, and have for a greater number of rays the properties possessed by the principal planes for paraxial rays. For there will usually be astigmatism in the image of a surface through a principal point; in fact, the only way in which astigmatism can be removed from the image it is desired to correct is to transfer it to the stop. The place of each principal plane must, therefore, be taken by two surfaces, one of which will relate to rays in a primary, and the other to rays in a secondary plane. The surfaces in the general case will only relate strictly to rays coming from a restricted axial region, and will alter in shape with changes in the part of the axis from which the rays considered arise, just as the real image of an object in a lens not free from spherical aberration and coma will have varying focal surfaces as the stop is moved along the axis. The results that have been proved may then be stated in the form that the coma is determined by the shape of these principal surfaces, and coma is removed from a system by ensuring that these surfaces are suitably curved. This is clearly seen in the case of any thin system of lenses where the principal surfaces are free from spherical aberration, coma and astigmatism as well as from distortion; an increase in the curvature of each lens surface by the same amount R causes an increase in the curvature of each principal surface by $R\,(1+\varpi)$ in the same sense, where ϖ is the Petzval sum. This, combined with the fact that the difference in the curvatures of the principal surfaces is ϖ times the power of the lens, accounts for the prominent part taken by the Petzval sum in the theory of the aberrations of thin lenses.*

In one particular case, which is of frequent occurrence, there is no need to make any special calculation to determine the amount of coma. This is when the object is at infinity. The values of r and ρ are then equal for all rays and the secondary second principal surface is determined by the intersection of the paths of the incident and refracted rays. The constancy of the length intercepted on the refracted ray between the incident ray produced and the image plane is the condition for freedom from coma in the secondary plane. If this condition is fulfilled for all rays it is easy to see from the equation for y' that the corresponding condition for rays in an axial plane must necessarily be satisfied also. When the condition is not satisfied the displacement due to coma of rays in a secondary plane is

$$-\left(\frac{\rho'}{f}+1\right)y_s',$$

and due to rays in a primary plane is

$$-\left(1+\tan\theta'\frac{d}{d\theta'}\right)\left(\frac{\rho'}{f}+1\right)y_s',$$

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* *See* Proc. Phys. Soc., Vol. XXVII., p. 494.

where y_0' is the distance from the axis at which the image due to paraxial rays is formed, f is the focal length and ρ' is the length intercepted in the way just described on the refracted ray inclined at θ' with the axis. It is evident that when there is no spherical aberration and the condition $\rho'=-f$ is satisfied the second condition also holds. In this form the conditions are of value in the calculation of telescope objectives by trigonometrical methods. Interesting alternative forms for the total primary and secondary displacements are

$$-\frac{d(\rho'\sin\theta')}{d(\sin\theta')}\cdot\frac{y_0'}{f}\text{ and }-\frac{(\rho'\sin\theta')}{\sin\theta'}\cdot\frac{y_0'}{f},$$

where it will be noted that $\rho'\sin\theta'$ is the length on the image plane intercepted between the incident and refracted rays.

The conditions under which Conrady's statement is correct may be readily found from the relations that have been proved in the course of this investigation. Equation (2) shows that any equation which involves the sine ratio, but not its differential coefficient can only refer to secondary rays, and also that the sine ratio will only measure the magnification for secondary rays if either $a=0$ or $p=0$. In the former case the zone considered is free from spherical aberration, and in the latter case the effective stop is in the object space in the surface containing the radial focal lines for an object at infinity in the image space. When the stop is so placed the primary and secondary magnifications are

$$\frac{\mu\, d(\sin\theta)}{\mu'\, d(\sin\theta')}\text{ and }\frac{\mu\sin\theta}{\mu'\sin\theta'}$$

respectively, but this expression for the primary magnification will only apply to the case when there is freedom from spherical aberration and the stop is not in the focal surface described, if the further condition is satisfied that the value of the spherical aberration is stationary on passing through this zone.

Exact formulæ of the class to which Conrady's and Chalmers' results belong may be found from the general equations already established. Such formulæ are not as general as those of equations (13) and (14), since the co-ordinates of rays in the object space do not appear in them, and the stop must consequently be assumed to lie in the image space. Let the stop lie in a plane at a distance ξ from the image plane, the equation of this plane being therefore $x'=\xi$. Denote by g_1 and g_2 the primary and secondary magnifications of the image, so that—

$$g_1=g+\text{primary magnification due to coma},$$

and

$$g_2=g+\text{secondary magnification due to coma}.$$

Eliminate p between equations (12) and (13). Then—

$$\frac{y}{y'}\cdot b-1=\frac{a}{\rho'}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

or, if the longitudinal spherical aberration is denoted by A—

$$\frac{1}{g_2}\cdot\frac{\mu\sin\theta}{\mu'\sin\theta'}-1=\frac{A}{\xi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

showing that the result $g_2=\dfrac{\mu\sin\theta}{\mu'\sin\theta'}$ holds when either $A=0$, or $\xi=\infty$, as has already been pointed out. Equation (17) is the form Chalmers' condition takes when the assumption that his coefficients i_1, i_2, &c., are independent of the zone of the lens is eliminated.* The corresponding relation for the primary magnification is obtaining by writing $a+M'^2a'$, $b+M'^2b'$ and $r'/(1-M'^2)$ for a, b and ρ' respectively in equation (16). The substitution of $\sin\theta'$ for M', $A\sec\theta'$ for a, $\sin\theta/\sin\theta'$ for b, and $\xi\sec\theta'$ for r', leads at once to

$$\frac{1}{g_1}\cdot\frac{\mu}{\mu'}\cdot\frac{d(\sin\theta)}{d(\sin\theta')}-1=\frac{1}{\xi}\left(A+\sin\theta'\cos\theta'\frac{dA}{d\theta'}\right), \quad . \quad . \quad (18)$$

as the equation for the determination of g, the primary magnification. The conditions under which the primary magnification is represented by $\dfrac{\mu}{\mu'}\dfrac{d(\sin\theta)}{d\,(\sin\theta')}$ may be verified from this equation. The formulæ (17) and (18) will in a number of cases be more readily applied to the results obtained in tracing rays for the determination of spherical aberration than those given earlier in the Paper. They are not, however, immediately applicable to the case of an object at infinity, and are not so suggestive as are equations (13) and (14) of the means that

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* In Chalmer's notation the equation would be written

$$\frac{OO_1}{JJ_1}\cdot\frac{\mu_0\sin B}{\mu_2\sin A}-1=\frac{KI}{N_1K},$$

which is simply a rearrangement of the equation

$$JJ_1=OO_1\frac{\mu_0\sin B}{\mu_2\sin A}\cdot\frac{N_1K}{N_1I},$$

he has reached on the middle of p. 6 before introducing approximations.

should be taken to remove coma in excess of the limit that can be tolerated. It is important in applying all the results to make sure that correct signs are given to all the quantities. In the case which most frequently arises—

θ, ρ, r, π, s are positive, and

$\theta', \rho', r', g, g_1, g_2, \xi, x$ are negative.

The spherical aberration A has the sign usually adopted—viz., positive for under-correction, negative for over-correction.

It may be noted that when the stop is in the image space the comatic displacements in the image of rays through a symmetrically-placed circular zone of the stop will necessarily obey the circular law described on p. 293. For, if $P-N'^2Q$ and $M'N'Q$ are written for p and q respectively, equation (7) becomes

$$\left\{\frac{\rho'}{1-M'^2-N'^2}+a+(M'^2+N'^2)a'\right\}P+b+(M'^2+N'^2)b'=0,$$

showing that P, like all the other variables, is a function of $(M'^2+N'^2)$, and thus constant for the zone. The new form of equation (6),

$$(\rho'+a)\{(M'^2+N'^2)Q-P\}=b,$$

shows that Q also is constant for the zone. Equations (2) may now be written in the form—

$$y'-Y'=-y\rho'\left\{\frac{(1-N'^2)P}{1-M'^2-N'^2}-N'^2Q\right\}$$

$$z'-Z'=-y\rho'\left\{\frac{P}{1-M'^2-N'^2}+Q\right\}M'N',$$

which indicate displacements of the type considered. It is, therefore, unnecessary in this case to assume $q=0$. The generality of the results obtained by assuming $q=0$ is not, of course, affected since the displacements are determined by p for both $M'=0$ and $N'=0$.

If this substitution for p and q is made in (4) the result is—

$$P(b^3+b')+b^2c+d+Q\{b-(M'^2+N'^2)b^3-N'^2(b^2c+d)\}=0,$$

so that the corresponding incident rays will only intersect on a circle, if either $Q=0$ or $b^2c+d=0$. The latter alternative is evidently the condition for freedom from astigmatism in the image of the stop for pencils of rays meeting the axis near the origins of co-ordinates. It is interesting to notice that the condition for freedom from astigmatism in a zone should contain the square of the sine ratio with respect to the centre of the stop for rays through that zone.

Equations (13) and (14) show at once why in first order coma the displacement of the intersection point of rays in a primary plane is three times that of the point for rays in a secondary plane. For it may be assumed that to a first approximation the principal surfaces are planes normal to the axis. Let j be the common limiting value for ρ'/ρ and r'/r when the rays become paraxial. Then $\rho'/\rho = r'/r = j \sec\theta' \cos\theta$.

Also $\sigma = \mu'/\mu$ and $s = \frac{\mu'}{\mu} \sec\theta \cos\theta'$, so that for secondary planes,

$$\frac{y'}{y} = j\frac{\mu}{\mu'} . \sec\theta' \cos\theta,$$

and for primary planes

$$\frac{y' - Y'}{y} = j\frac{\mu}{\mu'} . \sec^3\theta' \cos^3\theta,$$

or, since θ and θ' are small,

$$\frac{y'}{y} - j\frac{\mu}{\mu'} = \tfrac{1}{2} j\frac{\mu}{\mu'} (\theta'^2 - \theta^2)$$

for secondary rays, and

$$\frac{y' - Y'}{y} - j\frac{\mu}{\mu'} = \frac{3}{2} j\frac{\mu}{\mu'} (\theta'^2 - \theta^2)$$

for primary rays.

The connection which has been shown to hold between the coma of a real image and the curvature of the surfaces of constant magnification close to the stop is only one example of numerous reciprocal relations which hold for all lens systems. Another one has been mentioned incidentally in the course of the proof, viz., the necessity of transferring to the stop the astigmatism it may be desired to remove from the image for which the system is to be corrected. Other examples which are more obvious are the relation between the coma in the stop and the distortion of the image, and the spherical aberration in the stop and the change of distortion as the magnification is varied. The particular instance which forms the subject of this note is dealt with at some length because a knowledge of the results here established is likely to be of use to optical computers at the present time.

ABSTRACT.

The absence of coma from an optical system free from spherical aberration can be established by means of the sine condition, from the refraction of rays which originate at the centre of the object. In actual systems spherical aberration is always present to some extent, and the conditions which have to be secured in practical cases do not involve mathematical freedom from spherical aberration and coma, but rather their confinement within predetermined limits. Previous investigations on this subject have been published by Conrady and by Chalmers; the present Paper shows that the conclusions of both require some qualification. Conrady's conclusion, that the sine ratio for any zone gives the exact magnification, holds only for rays in a secondary plane, and one of the further conditions (*a*) that the zone is free from spherical aberration, or (*b*) that the stop is at the principal focus for rays travelling in the negative direction, must also be satisfied. Under these conditions the magnification for rays in a primary plane is $\left(1+\tan\theta'\frac{d}{d\theta'}\right)$ times that for the secondary rays, where $2\theta'$ is the angle subtended by the zone at the centre of the image. The conditions assumed by Chalmers involve the satisfaction of a relation between various quantities, including the curvature and astigmatism of the system; his results may in consequence not apply strictly where very large apertures are involved. Analysis of the general problem shows that the primary and secondary comatic displacements can be accurately derived from the properties of thin oblique pencils of rays. If a central incident ray inclined at θ to the axis is met by an oblique ray in the same axial plane at a distance r from its point of origin, and is refracted at an angle θ' with the axis and meets the image plane at a distance r' from the image of its intersection with the oblique ray, the primary comatic displacement of the oblique ray plus the aberrationless first order displacement is proportional to $r' \sec\theta'/sr \sec\theta$, where $\mu s/\mu'$ is the magnification in a primary plane for a small normal object at the point of intersection of the two rays. The corresponding secondary displacement is $\rho'/\sigma\rho$, where the Greek letters have the same meaning for secondary rays as the corresponding Roman letters for primary rays. These two quantities determine the coma exactly whatever the spherical aberration may be. It follows that coma is dependent upon the principal surfaces of a lens. When the object is at infinity the coma is determined by the locus of the point of intersection of a ray, incident parallel to the axis, with its refracted portion.

The results established in the Paper form one of a series of reciprocal relations which exist between the aberrations of an object and those of the effective stop.

DISCUSSION.

Mr. S. D. **Chalmers** pointed out the exact relation which he had given in the Paper quoted by Mr. Smith, and said that an assumption was necessary to apply this to the evaluation of coma. The assumption was that the two rays from an axial point and a neighbouring point to the secondary point on the stop would intersect in the image space substantially on the image of the stop. He thought this assumption was justified, though in exceptional cases it might require careful examination. Apart from such special cases he thought it was perfectly justifiable to use the expansion for the coma in powers of sin A, and to deduce the values in the primary plane from these in the secondary plane. It would be necessary to determine the values for sufficient apertures in the secondary plane to find each of the terms in the coma expansion.

Prof. F. J. **Cheshire** said he was pleased to see that the subject of optical designing was occupying an important place in the work of the Physical Society. The maker who used to turn out the best microscope objectives in Germany would sit with his microscope beside him and a large number of component lenses from which he would try one combination after another until a satisfactory result was reached. In this way some excellent objectives and a large number of mediocrities were obtained. Abbe set out to kill this trial and error method, and introduced at the Zeiss works the system of determining mathematically the complete structural data beforehand. In this country a trial and error method was still largely employed. An approximate design was computed and a trial instrument was made. This turned out badly, and the designer had another try, and so on. First-class results were impossible on these lines. It was true that we had two or three first-class designers in the country whose designs could safely be made up without trial; and this must ultimately be aimed at by every optician if we are to take a prominent position in optical work.

Mr. **Blakesley** said that an idea was involved in Mr. Cheshire's remarks which he had long held—viz., that a lens should only be used to form an image in the one particular position for which it is designed.

Mr. T. **Smith**, in reply to Mr. Chalmers, said that the exact relation mentioned led at once to an expression equivalent to the alternative formula (17). If expressed in this form the equation was exact. The transformation given in Chalmers' Paper, which involved the neglect of a term, was apparently made to enable the primary coma to be found on the supposition that expansion in terms of sin A would separate out the coma of different ratios. Such a method could be applied provided the spherical aberration were also analysed into terms of different orders in sin A, but not otherwise. The process would then be a comparatively laborious method of reaching the result given immediately by either of the formulæ for the primary comatic displacement contained in the present Paper. The formula given by Chalmers was correct in the absence of spherical aberration; if the spherical aberration of the first n orders were negligible, no serious errors in the estimation of the first $(n+1)$ orders of coma from his formula would arise, but appreciable spherical aberration of the nth order would throw out the estimate of coma of the $(n+1)$th and subsequent orders.

XXIII.—*Chromatic Parallax and its Influence on Optical Measurements.* *By* J. GUILD, *A.R.C.S., D.I.C., F.R.A.S.* (*From the National Physical Laboratory.*)

RECEIVED MAY 24, 1917.

§ 1. Introductory.
§ 2. Internal Chromatic Parallax.
§ 3. External Chromatic Parallax.
§ 4. Distinction between Chromatic and Positional Parallax.
§ 5. Constancy of TotalChromatic Parallax with Fixed Stop.
§ 6. Case of Cross Lines in a Dichromatic Field.
§ 7. Parallax in Optical Instruments.
§ 8. Methods of Avoiding Chromatic Parallax.
§ 9. Magnitude of Errors Involved.
§ 10. Possibility of Correcting Chromatic Aberration of the Eye.

§ 1. *Introductory.*

THE purpose of this Paper is to discuss the origin of the well-known difficulties involved in making satisfactory measurements of the position of coloured lines or bands in the blue and violet regions of the spectrum. As an *ab initio* discussion of the phenomena, as they are met with in practical cases, would be complicated by the properties of the particular optical instruments in question, it will be desirable to discuss first of all some simple results, which, though of no practical importance in themselves, are nevertheless interesting, and involve nothing but the essential elements of the problem.

That the human eye is only very imperfectly corrected for chromatic aberration is well known; but the important part which this defect plays in many phenomena of vision is not so generally understood or suspected. In the following paragraphs some cases of parallax arising from this cause will be discussed, and their serious effect in diminishing the ease and accuracy of certain optical determinations will be demonstrated.

The chromatic properties of the eye are generally very similar to what they would be if its contents were replaced by water.* Usually the actual curves connecting focal length and wave-length are slightly flatter, from $\lambda=520\ \mu\mu$ to $\lambda=660\mu\mu$; but in some cases they are difficult to distinguish from the water curves. For a general treatment of the subject it will be fairly accurate to regard the eye as a single spherical surface with the refractive and dispersive properties of water. Thus, while the focus for red rays is slightly longer than for the yellow-green, the shortening in the blue and violet is very marked. In ordinary vision in white light the effect of this is not noticed, as the purple halo surrounding the image of each point is swamped by the light from adjacent points. It is when the eye is functioning under abnormal conditions of aperture, illumination, &c., as is usually the case in making optical measurements, that the phenomena to be described become noticeable.

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* P. G. Nutting, Proc. Royal Society, XC., p. 440.

§ 2. *Internal Chromatic Parallax.*

In Fig. 1 (*a*) let *O* be an object point simultaneously emitting (or reflecting) two monochromatic radiations—*e.g.*, one blue and one red—and let the optical system of the eye be represented by a single refracting surface of which *B* and *R* are the points conjugate to *O* for the two radiations.* In the diagrams the line of sight is made to coincide with the optic axis of the system, since the obliquity of 5 to 7 deg. which actually exists does not materially affect the phenomena.†

In general, the retina will lie somewhere between the blue and red images, *B* and *R*, its actual position depending on the remoteness of *O* and the amount of accommodation which is being brought into play. When the object is seen most distinctly, the position of the retina will be as shown in the diagram, where the blue and red diffusion patches are of equal size. So long as the pupil is filled with light there will be no relative displacement of the diffusion patches, however the eye may be moved; for any motion of the eye can be represented by a simple rotation of Fig. 1 (*a*) about the point *O*, to which vision is directed.

In Figs. 1, (*b*) and (*c*), the effect of limiting the cone of light by a "pinhole" of 1 mm. to 2 mm. diameter, opposite either extremity of the pupil, is shown: the rays converge to *B* and *R* as before; but the restricted pencils strike the retina in separate narrow patches at *b* and *r*. Thus, two separate "images," one red and one blue, are seen. Moreover, since the diffusion patches at *r* and *b* are of small diameter, the

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* Of course, *O* is supposed to be at a distance from the surface large compared with the distances of *B* and *R*, which are, therefore, nearly coincident with the foci for these colours.

† See, however, reply to discussion.

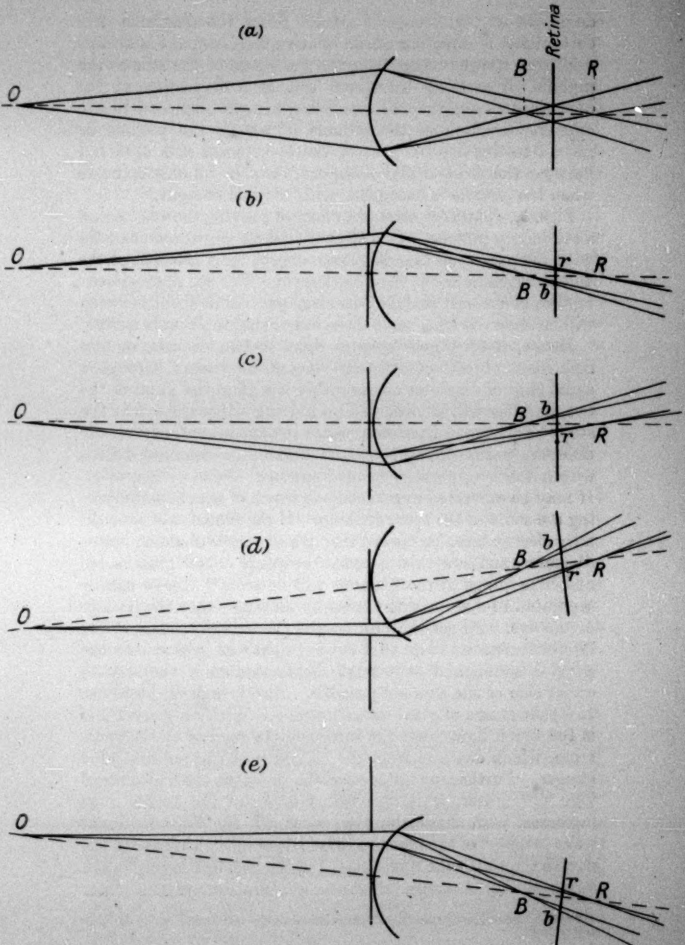
(a)
Retina
O
B
R
(b)
O
r
R
B
b
(c)
O
B
b
r
R
(d)
O
B
b
r
R
(e)
O
r
R
B
b

Fig. 1.

definition of each image is good. From the diagrams it is clear that the direction of the relative displacement is always such that the red image is on the same side of the axis as the pinhole. The mind interprets this as a separation of the object into two, the red one being always displaced in the *opposite* direction to the pinhole. Thus, if the pinhole be moved to and fro in front of the eye, parallax is observed between the red and blue " objects," which will only coincide when the pinhole is concentric with the line of sight.

Figs. 1, (*d*) and (*e*), show the effect of moving the eye behind a stationary pinhole. It is clear that these correspond exactly with the preceding two diagrams, except for a rotation of the whole diagrams about the object point. For the same *relative* motion of eye and pinhole, the same parallactic displacements will be observed whether it is the eye or the stop that is moved.

Hence, if an object seen in light of two colours, or two coincident objects of different colours, be viewed through a small stop of diameter considerably less than the pupil of the eye, parallax will be observed on moving either the eye or the stop. This type of parallax, since it depends entirely on the different paths pursued by the differently coloured beams within the eye, may be termed *internal chromatic parallax*. It may be exhibited in a variety of ways, of which the following are some of the more striking. If the filament of an ordinary electric lamp be viewed through a pinhole of about 1 mm. diameter, and two thicknesses of ordinary cobalt glass, a red and blue image of the filament will be seen.* These can be separated to a surprising extent by moving either the pinhole or the eye. To get the best results the experiment must not be performed in a room that is too bright; as in that case the pupil is small, and only small displacements of the stop to either side of the axis are possible. Also it is desirable to use two thicknesses of glass instead of one. With only one there is too much light from the intermediate regions of the spectrum, which detracts from the clearness of the red and blue images. Further, on account of the fact that the visibility of light diminishes, while the transmission of the cobalt glass increases, with diminishing wave-length, the " predominant wave-length " of the blue image is further in the violet with the double than with the single glass and the parallax is very much augmented. A single thickness of Wratten gelatine filter

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* Cobalt glass has a transmission maximum in the red as well as in the blue and violet.

No. 35 does as well as the double cobalt glass; while this filter, in addition to one cobalt glass, is still more satisfactory.

Similarly, if the capillary of a hydrogen vacuum tube which gives a good G' line, or a slit illuminated by a mercury lamp, be looked at through a pinhole, parallax effects are observed between the blue lines and the other strong lines of the spectra. In fact, the eye working under these conditions acts as a spectroscope of which the angular dispersion depends on how far from the axis the small incident pencil passes.

Another illustrative experiment is to illuminate a slit with deep blue light (a monochromatic illuminator is very suitable) and examine it through a pinhole. On moving the latter, or the eye, the luminous strip leaves the slit (which is itself seen in the general illumination of the room) and oscillates to either side of it in a most striking manner. During these excursions the vacated slit appears to be quite black, as though no light were transmitted by it at all.*

§ 3. *External Chromatic Parallax.*

In the cases considered in the preceding paragraph, the two differently coloured incident pencils originated at the same object or at objects equidistant from the eye We shall now consider the case in which the objects are so arranged that both images are formed at the same distance within the eye, and may, in consequence, be simultaneously focussed on the retina, as in Fig. 2 (*a*). If both objects are in the line of sight the images r and b will coincide. Since rays from all parts of the pupil converge, for each object, to the same point on the retina, no parallax can be produced by limiting the vision by an eccentric stop, as shown in Fig. 2 (*b*). In this case, therefore, moving the pinhole across the eye causes no apparent displacement of the objects.

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* In a recent Paper on "Visual Diffusivity" Phil. Mag., 33, pp. 18, January, 1917), H. E. Ives makes a casual reference to the danger of performing the experiments described in his Paper with a fixed stop in front of the eye on account of the apparent movements due to the chromatic aberration of the latter. In Prof. Gotch's "Purple-glass Test" (Burch's Physiological Optics, p. 48) the relative movements of the red and blue images of an electric filament, when observed through a pin hole and purple glass as described above, are employed to find the position of best focus for white light of an eye which has lost the power of accommodation. These are the only two cases of which the author is aware in which the effects of a relative motion of the eye and a small stop have been indicated, although the similar phenomena observed on eclipsing a portion of the pupil with the edge of a card or knife are described in many text books. The author is indebted to Mr. S. D. Chalmers for sending him the reference to Gotch's test after the reading of the Paper.

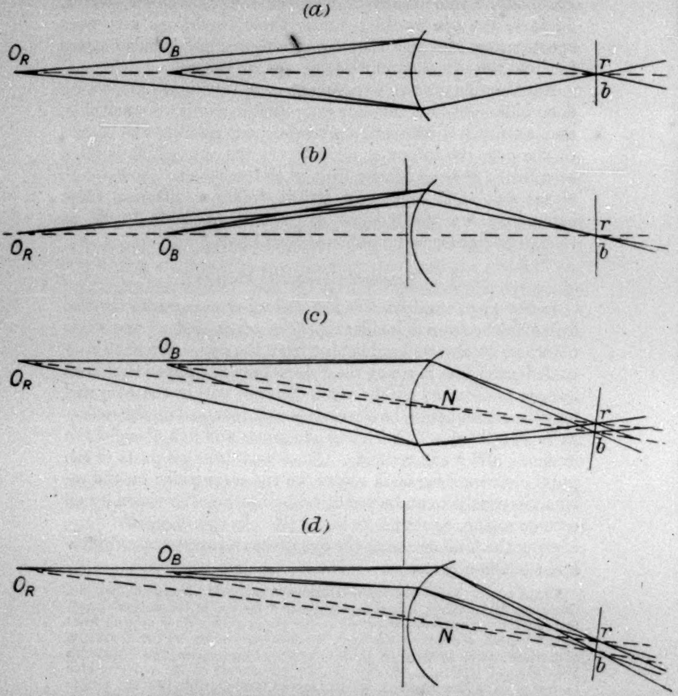

Fig. 2.

In Fig. 2 (*c*) the effect of a small displacement of the eye (with full pupil) is depicted The two images are still formed on the retina, but each must also lie on the line joining the corresponding object to the nodal point of the eye. A parallax, *r b*, is therefore the result. Here, again, since all the red rays intersect the retina at *r*, and all the blue rays at *b*, the interposition of a stop, as shown in Fig. 2 (*d*), will not affect the parallax; which, in this case, depends wholly on the angle subtended by the objects at the nodal point, and not on what part of the pupil is being employed.

This type of parallax, observed between two objects at different distances from the eye, *but of which the images are both focussed on the retina at the same time*, may be termed external chromatic parallax to distinguish it from the type previously discussed.

§ 4. *Distinction between Chromatic and Positional Parallax.*

At first sight it may not be obvious that the parallax described in § 3, and termed external chromatic parallax, has anything to do with the colour properties of the eye, since it depends only on the angle subtended by the objects at the nodal point. The essential feature of this parallax is, however, that it is unaffected by the interposition of a small stop, and is, therefore, observed even when such a stop is situated on the line joining the objects, as shown in Fig. 2 (*d*). Further, it is not observed on moving the stop in front of a stationary eye. In the case of the parallax ordinarily observed between two non-coincident objects seen in light of the same wave-length (or same predominant wave-length), the effect of a stop is exactly the reverse of this. It is clear that if we have a small fixed stop placed in line with two similarly illuminated objects, no parallax can be observed on moving the eye behind it; because, since the incident narrow pencils are concentric, the refracted pencils, being equally refracted, must also be concentric, and will, therefore, intersect the retina in concentric diffusion patches, however far apart the objects may be. This case is represented in Fig. 3 (*a*) Fig. 3 (*b*) shows the effect of an eccentric stop on the apparent position of two objects situated in the line of sight and which would appear coincident if seen with the full pupil. There is an apparent displacement, due to the separation of the points i_1 and i_2 in which the refracted pencils, converging to the true images I_1 and I_2, meet the retina. The displacement depends on the distance of the stop from the axis, and so parallax is observed on moving the stop across the eye, even if the latter is stationary. Thus, in the ordinary case, parallax is due to a change in the relative direction of the mean rays of the incident pencil which reach the eye. In normal vision this is the effect of moving the eye; but if the fulfilment of this condition is prevented, as by the interposition of the fixed pinhole, moving the eye will not produce parallax; or, if the condition is fulfilled without moving the eye, as by moving the pinhole, parallax will occur.

There is thus complete dissimilarity in the essential features of external chromatic parallax and that due simply to the difference in distance of two similarly illuminated objects from the eye.

The fundamental properties of the three types of parallax which may be encountered in suitable circumstances may be summarised as follows :—

Positional Parallax.—Object planes separate : image planes also separate : objects similarly illuminated : parallax produced if eye is moved with full pupil operative : or if a small stop is moved in front of stationary eye : not produced if eye is moved behind a fixed stop.

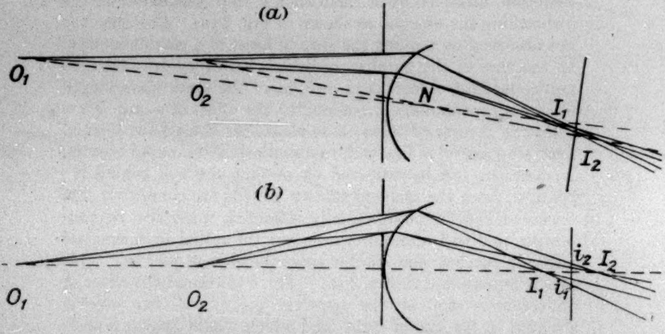

FIG. 3.

Internal Chromatic Parallax.—One object plane : two image planes : objects seen in light of different wave-lengths : parallax produced by any relative motion between the eye and a small stop. It cannot be observed with a full pupil.

External Chromatic Parallax.—Two object planes : one image plane : objects seen in light of different wave-lengths : parallax produced on moving eye either with or without a stop in front of it : not produced on moving the stop in front of the stationary eye.

§ 5. *Constancy of Total Chromatic Parallax with Fixed Stop.*

If neither the object planes nor image planes are coincident in the case of two differently coloured objects, the parallax observed on moving the eye behind a fixed stop will be neither internal or external in character, but will be compounded of both. As this is the most general case, it may be worth while investigating it quantitatively. The approximations which are made in the following treatment are self-evident :—

Let R and B, Fig. 4, be two objects (red and blue respectively) situated near the line of sight. Let R' and B' be the points in which the axis is intersected by RD and BD, the principal rays of the thin pencils which enter the eye through the small stop at D. The principal rays of the refracted pencils must cross the axis at R'' and B'', the points conjugate to R' and B'. The points r and b, where these rays meet the retina, are the centres of the observed images ; and the angle which rb subtends at the posterior nodal point N_2 is the apparent angular separation of the objects.

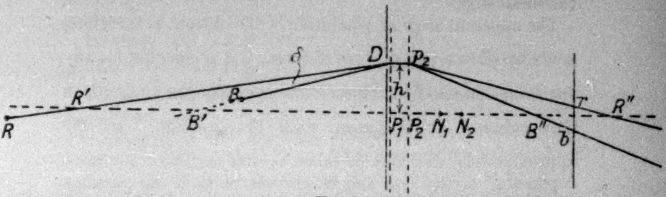

FIG. 4.

In the figure, the blue object is nearer the eye than the red one ; but is still so remote that the blue image is anterior to the red image. This is, therefore, a case intermediate between internal and external chromatic parallax. The analysis holds, however, for any order of objects and images.

P_1, P_2, N_1 and N_2 are the principal and nodal points of the system. Denote P_1R' by U_r ; P_1B' by U_b ; P_2R'' by V_r ; and P_2B'' by V_b. Let μ_r and μ_b be the refractive indices of the final medium for the two wave-lengths, and μ its index for the mean rays of the spectrum (say sodium light), and let f_r, f_b and f be the corresponding values of the first principal focal length.

Then, from the general formula $\frac{1}{U}+\frac{\mu}{V}=1/f$, we obtain

$$\frac{1}{V_r}=\frac{1}{\mu_r}\left\{\frac{1}{f_r}-\frac{1}{U_r}\right\} \text{ and } \frac{1}{V_b}=\frac{1}{\mu_b}\left\{\frac{1}{f_b}-\frac{1}{U_b}\right\}.$$

The angle $rN_2b \doteq \mu \times$ angle $rP_2b \doteq \mu \times$ angle rp_2b,

$$=\mu\left\{\frac{h}{P_2B''}-\frac{h}{P_2R''}\right\}=\mu h\left\{\frac{1}{V_b}-\frac{1}{V_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b}\left(\frac{1}{f_b}-\frac{1}{U_b}\right)-\frac{1}{\mu_r}\left(\frac{1}{f_r}-\frac{1}{U_r}\right)\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}+\frac{\mu_r U_b-\mu_r U_r}{\mu_b\mu_r U_b U_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right\}-\frac{\mu}{\mu_b}\left\{\frac{h}{U_b}-\frac{h}{U_r}\right\}$$

$$=\mu h\left\{\frac{1}{\mu_b f_b}-\frac{1}{\mu_r f_r}+\frac{1}{U_r}\cdot\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right\}-\frac{\mu}{\mu_b}\delta,$$

where δ is the angle subtended by the objects at the centre of the small stop.

The apparent angular separation of the objects is, therefore, made up of two parts. One of these, $\frac{\mu}{\mu_b}\delta$, is constant for any one arrangement of the objects, and is simply the angle which they subtend at the stop, since the fraction $\frac{\mu}{\mu_b}=0{\cdot}995$ approximately, and may be taken as unity. The other part, depending on the sign and magnitude of h, is the parallax obtained by moving the eye behind the stop, a distance h off the axis. It would clearly vanish, for all values of h, if the system were achromatic, or if both wave-lengths were the same; and is, therefore, wholly chromatic in origin; bearing out the result arrived at in § 4 that a fixed stop prevents the observation of purely positional parallax, so long as all the light passing through the stop enters the pupil.

Since $\quad \frac{1}{f_r}\doteq\frac{\mu_r-1}{\mu-1}\cdot\frac{1}{f}$; and $\frac{1}{f_b}=\frac{\mu_b-1}{\mu-1}\cdot\frac{1}{f}$,

the expression for the parallax may be written

$$\mu h\left\{\frac{1}{(\mu-1)f}\left(\frac{\mu_b-1}{\mu_b}-\frac{\mu_r-1}{\mu_r}\right)+\frac{1}{U_r}\left(\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right)\right\},$$

which reduces to

$$\mu h\left(\frac{\mu_b-\mu_r}{\mu_b\mu_r}\right)\left(\frac{1}{(\mu-1)f}+\frac{1}{U_r}\right).$$

To obtain an approximate numerical estimate of this quantity we may assume the refractive properties of the living eye to be those of distilled water about 40°C.

These are

$$\mu_C=1{\cdot}32878\,;\ \mu_D=1{\cdot}33061\,;\ \mu_F=1{\cdot}33473\,;\ \mu_{G'}=1{\cdot}33793.$$

If we take C and G' as the red and blue wave-lengths respectively, we get for the parallax*—

$$\pi=h\times1{\cdot}33\left(\frac{0{\cdot}00915}{1{\cdot}3288\times1{\cdot}3379}\right)\left(\frac{1}{0{\cdot}3306}\cdot\frac{1}{17}+\frac{1}{U_r}\right)$$

$$=h\times0{\cdot}00685\left(0{\cdot}178+\frac{1}{U_r}\right).$$

The coefficient of h is constant except for the effect of variations of U_r. If this varies from ∞ to 250 mm.—the near point for normal eyes—the factor in brackets only ranges from 0·178 to 0·182, a variation of a little over 2 per cent. Thus, to within a small percentage, the parallax between two differently coloured objects, seen through a small fixed stop, is independent of the position of the objects.

Its value may be taken as—

$$\pi=h\times0{\cdot}00685\times0{\cdot}178=h\times0{\cdot}00122 \text{ radians}=h\times4{\cdot}2 \text{ minutes}.$$

With a total motion of the pupil of 5 mm. to 6 mm., a parallactic range of 20 to 25 minutes should be encountered. The full range which the author has observed in various optical instruments lies between 15 and 27 minutes, when corrected for magnification; depending on the size of the pupil at the time. This indicates how closely the assumptions made as to refractive properties fit the actual case.

§ 6. *Case of Crosslines in a Dichromatic Field.*

The results of the preceding paragraphs concerning coloured objects are clearly applicable to the case of dark objects seen against coloured backgrounds. An important case of this type is that of the crosslines of optical instruments.

To study the properties of crosslines under various conditions of illumination the arrangement of Fig. 5 is very convenient. A transparent grating G, is mounted between two lenses

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* The mean focal length of the eye is here assumed to be 17 mm., the figure given in Tscherning's Optique Physiologique. In the schematic eye the focal length is 15·5 mm.; while in the "reduced" eye of Donders it is only 15 mm. Either value would give a greater figure for the chromatic parallax than the one adopted.

L_1 and L_2 to act as collimator and telescope objectives respectively. Two gas-filled half-watt tungsten lamps, with short straight lengths of spiral filament, are mounted on stands so that the first order spectra on opposite sides of the normal, due to the two sources, are focussed at *P*, the observing pinhole An eye placed behind *P* sees the back of the lens uniformly illuminated by a mixture of the two wave-lengths which are focussed at *P*. By adjusting S_1 and S_2 any two wave-lengths can be combined. A rheostat in each lamp circuit enables the relative intensity to be varied.

The crosslines *W* of thin copper wire are mounted on a diaphragm, and placed in front of the lens L_2.*

If we commence with a monochromatic blue field, and add a gradually increasing proportion of light of some longer wave-length, say green or red, we find when the proportion of this colour reaches a certain value that two sets of wires are visible, which exhibit parallax on moving the eye behind the stop. As the second colour increases in intensity the two sets become equally well marked. Finally, when the longer wave-length predominates strongly, only the second image can be seen, the original one being blotted out. If the added illumination is white light, obtained by placing one of the lamps, so that its direct image is formed at *P*, the result is similar to that described; since white light, for focal purposes, and consequently for parallax effects, is equivalent to a monochromatic illumination of wave-length somewhere in the yellow green, depending on the nature of the source. In general, these effects will be produced if the blue field is diluted with any light, monochromatic or mixed, of different predominant wave-length.

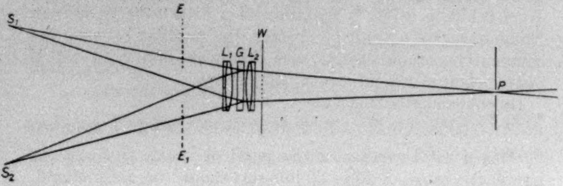

Fig. 5.

By interposing vertical screens, shown dotted at *E* and *E′*, each colour may be confined to half of the field; and the behaviour of cross lines, set on the edge of a coloured band, when the rest of the field is illuminated with light of different wave-length, may be observed. This consists of a breach of continuity of the lines at the junction of the colours, the parts of the cross on each side of the boundary separating or overlapping as the edge is moved. The relative motion of the fields themselves as well as the cross wires is very striking in the experiments with this apparatus. Further, when working with blue or violet fields, a third image of the aperture is seen due to the large proportion of undiffracted white light which is scattered by the rulings of the grating. This in nowise affects the relative movements of the *coloured* images, which behave quite independently. If both of the spectrum colours are blue, the scattered white light can be cut out by a cobalt glass filter if desired. With the arrangement employing the screens E and E', the difference in wave-length required at any part of the spectrum, before appreciable parallax effects are introduced, was determined. One half of the field was illuminated with light of a certain wave-length, and, by moving the appropriate lamp, the wave-length of the other half was adjusted until no discontinuity of the cross-lines could be produced by moving the eye. The results showed that at $\lambda=410\mu\mu$ the difference of wave-length, in the direction of increasing λ, at which parallax could just be detected was only $5\mu\mu$; at $\lambda=430$ this had increased to $10\mu\mu$; at $\lambda=450$, to $15\mu\mu$; at $\lambda=500$, to $20\mu\mu$. It is difficult to detect parallax between two colours of wave-lengths greater than $560\mu\mu$ until one of them reaches the extreme red. The above results apply, of course, only to the author's eyes, but they are probably typical of most normal cases.

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* The crosslines should be of the diagonal type; not vertical and horizontal.

§ 7. *Parallax in Optical Instruments.*

A large proportion of the work in practical optics involves the determination with the highest possible precision of the position of spectrum lines, or the edges of coloured bands, in the field of view of a telescope. In properly designed instruments, in order to make full use of the available resolving power, the magnification of the telescope employed is usually from 20 to 25 linear, for each inch of effective aperture of the dispersive system. A real image of the face of the last prism, or whatever limits the effective aperture of the beam entering the object glass, is formed by the remainder of the optical system, and can be seen just in front of the eyepiece, if the eye is held about 9 in. away. All the light passing through the system must pass within the limits of this image, which is termed the "exit pupil" of the instrument; and which acts as a fixed stop through which vision of the field of view takes place. For the whole of the field to be seen, the pupil of the eye must be coincident with, or at least somewhere near, the exit pupil of the telescope. The diameter of the exit pupil in any direction is equal to the diameter of the effective aperture in the same direction divided by the magnification of the telescope. Its horizontal diameter is never, therefore, much more than 1 mm. in instruments of precision It follows from this that ordinary parallax, due to want of simultaneous focus of the cross lines and objects in the field of view, can never be observed in such instruments. The following experiment illustrates this: set the cross lines of a spectroscope, such as a Hilger constant deviation instrument, on a bright yellow or green line, using a fairly wide slit, so that there is light enough from the line itself to see the cross lines by. It will then be found, however far out of focus the line may be, that there is no parallax observed on moving the eye, except for spasmodic jerks at the very extreme range of the motion, due to the fact that at these points the edge of the iris is eclipsing part of the exit pupil, and thereby altering the relative direction of the principal rays of the effective pencils; which, we have seen, is the essential condition for positional parallax: but so long as the position of the eye permits all the light transmitted by the exit pupil to enter it, there is no parallax.

If, however, the cross wires are illuminated with light of different refrangibility from the line or band under observation, chromatic parallax effects will be observed; and these, as shown in § 5, cannot be eliminated by varying the focus either of cross wires or spectrum lines. Unfortunately, this is almost always the case, from a variety of causes, when measurements are being made in the blue and violet part of the spectrum. In the case of instruments such as the ordinary spectrometer, when spectrum lines are reduced to the fineness essential for accurate work, the cross hairs are usually quite invisible; and it is necessary, before settings can be made, to supply some general illumination in the field of view. This, as far as the author's experience goes, is almost invariably accomplished by arranging a table lamp so that it throws sufficient light into the telescope to render the cross hairs distinctly visible; the result being that large parallax effects are produced with blue and violet lines. In some cases where there are many intense lines in the most luminous part of the spectrum, there may be sufficient illumination, due to the light scattered at the surface and within the material of the lenses and prisms, and also to light of other colours reflected from the sides of the tubes, to make the cross wires visible in the blue and violet regions, without the addition of extraneous light. The result, however, is the same, since the scattered light is undispersed and of quite different predominant wave-length from the lines under examination.

The serious hindrance which this puts in the way of accurate observations on lines of short wave-length is one with which most optical experimenters are familiar. The obvious result is that much greater variation of individual readings is encountered than when working in the yellow or green; and correspondingly longer series of observations must be made. There is, however, a more serious source of uncertainty, which patience in taking a multiplicity of readings will not eliminate. An observer, when using only a limited portion of his pupil, has a tendency to place his eye in one position more than in others; the favoured position being that in which the small pencil of light encounters fewest optical imperfections in its way to the retina, and produces, in consequence, the best image. The factors which determine the path of best definition will be the existence of any of the imperfections grouped under irregular astigmatism; and also the temporary arrangement of "muscæ volatantes," striæ in the humours, &c. Such differences will be slight, and individual placings of the eye will vary considerably; but the mean to which the observer's settings will always approximate, if sufficient readings be taken, will be that corresponding to the favoured position of the eye. In general, this favoured setting will not be that in which the principal rays pass through the nodal points of the eye, and there will, therefore, be a difference in the ultimate result which the observer obtains under these conditions from that which he would obtain were chromatic parallax absent. Moreover, this error varies from time to time, probably on account of variation in the distribution of the temporary irregularities mentioned above, and possibly others. It is therefore impossible to determine a reliable correction to be applied to readings made by a particular observer under specified conditions.

Before proceeding to deal with the simple methods which can be employed to obviate all difficulties due to chromatic parallax, reference may be made to the erroneous method which students of practical physics sometimes attempt to employ for the focussing of telescopes—viz., to adjust until there is no parallax between the line, or other object in the field, and the cross wires. With very low-power telescopes, such as are sometimes found on instruments intended for the use of students, the process may meet with a limited amount of success; but the method is essentially wrong, and cannot be applied at all with telescopes of which the magnification bears the most suitable ratio to the aperture. In such cases the only criterion of exact focal adjustment is simultaneous sharpness of definition of object and crosshairs, and a critical judgment of definition is one of the most generally useful faculties that a student of optics can acquire.

§ 8. *Methods of Avoiding Chromatic Parallax.*

The essence of such methods clearly lies in supplying illumination for the cross wires of the same wave-length as the line on which they are to be set; and in excluding from the field all stray light of other wave-lengths if it is present to an appreciable amount. The most suitable method of fulfilling these conditions depends on the circumstances. If the wave-length concerned is not very far in the blue it will sometimes be sufficient to throw some white light from a table lamp into the field and employ a filter of cobalt glass in front of the eye-piece. This may bring the predominant wave-length of the field illumination sufficiently near to that of the line to destroy most of the chromatic parallax between it and the cross wires. At the red end of the spectrum a piece of red glass or, preferably, a Wratten gelatine filter, No. 25, mounted between glass, is all that is ever required, since the effects at this end are very much less marked than with short wave-lengths. For most of the blue and violet, however, the small difference in wave-length necessary to produce parallax, renders the cobalt glass practically useless. A method of general applicability for illuminating the cross hairs in instruments of the spectrometer type, is to arrange an auxiliary source of white light, so as to produce a continuous spectrum as a background to the line spectrum. By this means the cross wires are always illuminated by light of the same wave-length as the line on which they are set, and no parallax effects are present. The observations are then as satisfactory, other things being equal, as those made at other parts of the spectrum. A suitable arrangement is shown in Fig. 6. The line source S_1 is focussed on the slit by the lens L_1 in the usual way. The white light source S_2 is focussed on the slit via the 45 deg. reflector R, consisting of a piece of unsilvered microscope cover glass. A very intense source at S_2 is necessary in order, when using a fine slit, to get a sufficiently bright background as far in the violet as is sometimes required. If the lines are bright enough to stand some cutting down, the continuous spectrum can be greatly intensified by semi-silvering the reflector ; but it is better in this case to use for R a piece of thin mirror glass, from which the silver has been scraped off, in parallel strips of 1 mm. to 2 mm. broad. If the width of the silvered and unsilvered strips are equal, the reflected and transmitted intensities are nearly equal. Reflectors can be made giving any desired proportion to the intensities. With this type of reflector a satisfactory background can be obtained, well beyond the *H* and *Hg* lines at 434 and 436 $\mu\mu$, from a "Pointolite" tungsten lamp. To get much beyond this—to include, for instance, *Hg* 404—or, in cases where the lines are so faint that only the plain glass reflector can be employed, it is essential to use a strong carbon arc. If a strip reflector is used, it must be further from the slit than shown in the diagram, so that the strips are not visible in the field of view. The source S_2 must be shaded by a sheet iron or asbestos cover, which only lets light out in the required direction ;

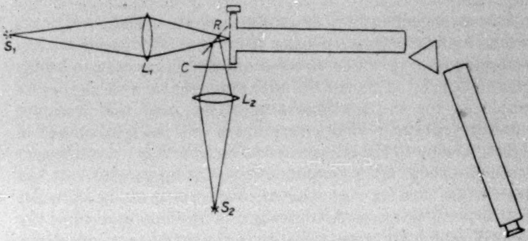

FIG. 6.

otherwise, the general blaze which it creates is very annoying. It is further desirable to insert a cobalt glass screen at C in order to cut out the most luminous constituents of the light scattered within the instrument. The illumination of the field due to scattered light is, of course, much the same at whatever wave-length the instrument is set, but it is quite swamped in the luminous parts of the spectrum by the relative brilliance of the spectrum colours. In the blue and violet, however, the luminosity of the spectrum colour is so low that the scattered light is relatively intense; and, in most instruments, illuminates the cross lines quite distinctly, even beyond the limits of the visible spectrum. If this stray white light is not absorbed by the filter C, the effectiveness of the spectrum illumination is impaired in the blue, and nullified entirely in the violet. A modification of the above arrangement is to employ a separate collimator for the white light spectrum, placing it so that the parallel beam emerging from it can be directed into the prism system by the glass reflector. This has the advantage that a wide slit can be employed for the continuous spectrum, and the necessary intensity obtained with a very much less powerful source of light. Owing to the adjustments required, it is a troublesome method, except for a permanently set up apparatus. It has the further disadvantage that the reflector must be optically flat on both sides, or the definition of the line spectrum, the light of which traverses it between the collimator and prism system, is spoilt.

While the above methods are probably the most satisfactory that can be employed where circumstances make it essential to employ fine lines, there is, nevertheless, a very much superior method of using a spectrometer, which is applicable to almost all purposes for which such instruments are now used. This consists in replacing the stereotyped fine slit by a wide slit, ·5 mm. to 1 mm. in width, with a central spider line mounted parallel to the jaws. Each spectrum line is represented in the field of the instrument by a broad coloured band with a fine black line down the middle. The cross lines are set on this line. This method has many advantages over the use of fine bright lines even under the most ideal conditions. In the first place it entirely prevents the occurrence of chromatic parallax without the necessity of employing auxiliary illumination for the cross lines; since the latter are seen sharply defined on the bright band of spectrum colour. Further, the comfort experienced in setting black cross hairs* on a black line under these conditions of illumination, is of the very greatest importance in maintaining the observer's efficiency during prolonged series of observations Everyone is aware of how, in setting on bright lines even under the best conditions as to illumination of cross lines, there is always a feeling of uncertainty when the setting is actually accomplished. Just before hand, when the cross hairs and line are distinctly separate, both appear ideally sharp, and the observer feels that superposition can be effected with the greatest accuracy; but when the superposition has actually taken place, diffraction effects make themselves evident in a wavering and loss of definition at the point of intersection, which produces an annoying lack of confidence in the accuracy of the setting, and is, in addition, very trying to the eyes. With the arrangement described there is no such effects at the moment of contact: everything appears as distinct and clear as if it were a diagram drawn on paper. Further, the tiresome fluctuations of accommodation, to which the eye is subject when using faintly illuminated cross lines, are practically absent.

The method has also the advantage that the full brightness of any line is utilised. With a slit cut down so as to give a fine line spectrum the intrinsic brightness of the lines is very much reduced. Since it is essential to keep the illumination of the cross wires faint enough in comparison with the line to preserve a satisfactory contrast, it is clear that with lines of low visibility the wide slit method gives a much more satisfactory working illumination, since, in that case the full brightness of the line is utilised in lighting the cross wires, instead of something which must be considerably less bright than the line, even with the latter at reduced intensity.

This method can be employed with advantage for all such purposes as the measurement of refractive index and dispersion on the spectrometer. The full separating power of the instrument is obtained, since the spider line is of negligible width. With sodium light, for instance, the broad yellow band of superposed D_1 and D_2 is traversed by the black lines due to the two components, each as clearly defined as if the other component were not there. Except with spectra of great complexity, such as that of the iron arc, the superposition of the bands of neighbouring lines does not give any trouble.

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* The author refers throughout to cross lines mounted diagonally. Unfortunately, there are still many instruments made in which the cross lines are vertical and horizontal. It is impossible to make accurate settings with such lines unless a slit of appreciable width be used and its image bisected by the vertical cross line. See, however, reply to discussion in connection with a setting of this type.

The same type of setting is suitable for use in the measurements of the angles of prisms; and here, as in the case of refracted lines, it gives a marked gain in comfort and accuracy compared with the stock method of using the image of a fine slit.

The spider line may conveniently be mounted permanently between thin glass plates, or the fine cross lines ruled on glass which replace spider lines in some instruments may be used. The author has sometimes employed for this purpose the finely divided micrometer scale—1 cm. divided to millimetres and tenths of millimetres—from a Zeiss microscope eyepiece. The scale is attached across the slit with the rulings parallel to the jaws. Each spectrum band is now crossed by an image of part of the scale, and a suitable graduation is chosen on which to make settings. On account of the extreme fineness of the rulings this proves very satisfactory.

Spider lines, while giving excellent results with instruments in which the definition is very good, must be replaced by something coarser when the lenses or prisms are not of the best quality, and in low-precision instruments generally. The author has found that two or three quartz fibres, of thickness varying from 5 to 20 thousandths of a millimetre, are suitable for most purposes. Even for the finest work there does not appear to be any great advantage in using a line finer than 7 or 8 μ; this, in fact, being a very convenient thickness to employ in work demanding the greatest precision. The plates between which the fibre is mounted should be separated by at least a millimetre to avoid interference effects, and the one next the slit should either be of optical glass or of thin microscope cover glass; otherwise the sharpest definition of the line will not be obtained. Two transverse fibres—or lines ruled on the nearest glass—should also be supplied. They should be about 0·5 to 1 mm. apart, and serve to indicate the middle of the slit. Adjustments should be made so that the cross lines in the eyepiece bisect the vertical interval between these transverse lines.

An instrument of great practical importance in which chromatic parallax is often extremely troublesome is the Pulfrich refractometer, which is at present employed in the commercial testing of optical glass. In the field of view of this instrument each spectrum colour gives rise to a more or less broad band, on one edge of which the cross lines have to be set. There is usually a very large proportion of stray light in the field, due to scattering of the incident beam by the matt surface surrounding the polished circle on which the specimen is placed. This illuminated surface can usually be seen sending light straight into the telescope if the eyepiece is removed. The defect is worst at the larger angles, owing to the diminution of the effective aperture of the polished surface with increasing obliquity. Some of this scattered light can be cut out by a judicious use of the revolving cap; but it is usually impossible, except within a limited range of angle, to avoid the presence of sufficient, from this cause alone, to produce serious parallax troubles with blue and violet lines.

With the hydrogen spectrum, as usually obtained from commercial tubes, the *G′* line is so faint that it is rarely possible to see more than one image of the cross lines, which moves about with respect to the edge of the band as the eye is moved. With a brighter band, obtained from a heavily run tube under proper conditions of pressure,* the various effects described in § 6 are usually obtained.

Unfortunately the nature of the optical system of the Pulfrich refractometer precludes the possibility of using the improved type of setting, which has been described for instruments of the spectrometer type. The best that can be done in this case is to interpose a piece of cobalt glass in front of the specimen to cut out the most luminous constituents of the stray light. It is not usually necessary to supply continuous spectrum illumination when using hydrogen, as the overlapping bands of the secondary spectrum supply sufficient continuous background to render the cross lines visible. The reduction in intensity of the *G′* band itself, due to the absorption of the cobalt glass, makes it essential to use very heavy discharges; but when this is done settings practically free from the disturbing effects of parallax, and usually as accurate as those on any other line, can be made. If it is necessary, in any case, to supply spectrum illumination for the cross lines, this can be effected by replacing the right angle prism for use with flames by a thin glass plate, and arranging a half-watt tungsten lamp so as to produce a continuous spectrum in the field of the refractometer. The brightness of this can be controlled by means of a rheostat in the lamp circuit, to give a suitable background for the cross lines at any part of the spectrum. The lamp should be shaded in all but the one direction to prevent glare.

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* *See* Proc. Phys. Soc., Vol. XXVIII., p. 60; or Collected Researches National Physical Laboratory, Vol. XIII., p. 53.

§ 9. *Magnitude of Errors Involved.*

Many sets of observations have been made by the author, and by several of his colleagues, to determine the errors liable to arise in using instruments of different types, if chromatic parallax is not eliminated. A brief summary of the results will suffice to show the seriousness of these. In § 5 it was shown that a total range of parallax of over 20 minutes in apparent angle might arise in certain circumstances from this cause. With an optical instrument this figure has to be divided by the magnification to obtain the equivalent range in the actual settings. For example, in a particular spectroscope, of the constant deviation type, furnished with a micrometer eyepiece, the extreme range of the settings that could be made on the *G′* line, with a reddish white background to the cross lines, was, with a dark adapted eye, 0·14 mm. on the micrometer or 1·8 minutes in angle. The power of the telescope was 15, so that the apparent angular range was 27 minutes. With white illumination of the field from a table lamp, about three-quarters of this range was obtained. In fact, the effect does not diminish greatly until the predominant wave-length of the field illumination becomes fairly near that of the line. When chromatic parallax was absent settings could be repeated to 0·001 mm. The maximum difference obtainable from the true readings with improper illumination was therefore, in this case, about 70 times the observational error under proper conditions. Of course, in actual determinations the eye would never be used near either extreme of the pupil. As an example of the discrepancies which may be encountered in practice, the following observations were made on the mercury line at λ, $436\mu\mu$. No adjustments were altered during the whole set of observations; which are, therefore, comparable one with another. The readings are the arbitrary divisions on the micrometer head, 0·1 corresponding to about 1 second. In series A, the field was illuminated with white light from a table lamp; in B the same lamp was used, but the light from it was passed through a Wratten No. 49 filter, which is better for this line than cobalt glass; and in C spectrum illumination was used. In each case the greatest care was taken to get as satisfactory settings as possible, and the cross wires were always moved aside, before applying the eye to the eyepiece, to prevent one setting from biassing the next. The table shows that in the first two series, in which chromatic parallax was present, the probable errors of individual settings were respectively seven and four times that of series C, in which it was not present. Further, the difference between the mean values of series A and B from that of series C—which may be taken as the true reading—shows the danger of relying on the mean, even of an extended series of observations.

Series A. White light in field.		Series B. Filtered light in field.		Series C. Spectrum illum. of field.	
Observations.	Numerical difference from mean.	Observations.	Numerical difference from mean.	Observations.	Numerical difference from mean.
17·4	0·2	18·4	0·7	17·4	0·1
16·2	1·2	17·7	0·0	17·4	0·1
17·5	0·1	16·8	0·9	17·2	0·1
17·1	0·5	17·4	0·3	17·3	0·0
15·9	1·7	17·5	0·2	17·3	0·0
18·0	0·4	17·8	0·1	17·2	0·1
19·1	1·5	17·7	0·0	...	...
18·4	0·8	16·9	0·8	...	...
18·	0·4	18·1	0·4	...	...
16·9	0·7	17·7	0·0	...	...
17·5	0·1	18·0	0·3	...	...
17·6	0·0	18·2	0·5	...	...
15·8	1·8	18·6	0·9	...	...
17·1	0·5	18·7	1·0	...	...
19·4	1·8	17·3	0·4	...	...
18·6	1·0	17·3	0·4	...	...
16·8	0·8	17·7	0·0	...	...
18·5	0·9	18·1	0·4	...	...
17·5	0·1	17·8	0·1	...	...
18·0	0·4	17·0	0·7	...	...
Means 17·6	0·7	17·7	0·4	17·3	0·1

The results show that the variations in individual settings are much reduced by the use of the filter, but that they are still several times in excess of what they should be.

Results of a similar character have been obtained for various types of instrument by the author's colleagues. For example, readings for the *G′* line, obtained on the Pulfrich refractometer by certain observers, sometimes showed persistent differences of eight or nine-tenths of a minute—a discrepancy at least five times as great as is tolerable in these determinations. Usually the agreement was much better, but it is only when the proper conditions of illumination are secured that results can be obtained of an accuracy comparable with that for other lines.

§ 10 *Possibility of Correcting the Chromatic Aberration of the Eye.*

By employing an eyepiece of suitable design, or by the provision of a suitable spectacle lens, the chromatic aberration of the eye could be corrected in the sense that differently coloured objects in the same object plane would have their images simultaneously focussed on the retina. It is not at present known, on account of insufficient data, to what extent vision would be improved by the employment of an achromatising lens in front of the eye ; or whether the definition of optical instruments, such as telescopes and microscopes, could be improved by introducing the chromatic properties of the eye into the computations ; but it is highly probable, on general grounds, that an appreciable improvement would result in many cases. At any rate, it is desirable that accurate information of the chromatic properties of the eye, and the range of variation met with in different persons, should be available. The author had commenced an investigation of methods suitable for such determinations over a year ago ; but the work is at present in abeyance for lack of time.

But whatever advantages might or might not accrue from some method of compensating the chromatic aberration of the eye, it is, unfortunately, the case that no such method is likely to obviate the difficulties from this cause which have been discussed in the preceding paragraphs. Let the eye-piece be so designed that coincident objects of different colours in the field of view are finally focussed on the retina. This is only accomplished by altering the positions of the virtual objects with respect to the eye ; and this, as we have seen, has no effect on the parallax. If, on the other hand, we were to achromatise the eye by means of a spectacle lens, fixed to the head by the ordinary framework, and partaking of all the movements of the eye, the essential condition for absence of parallax—viz., that the moving system shall be achromatic in itself—would be fulfilled ; but the achromatising lens would require to be exactly coaxial with the eye, otherwise parallax errors would result ; for it is possible to arrive at the condition in which the lens and eye are not coaxial, by starting from the condition in which they are coaxial and moving the eye while the lens remains fixed. During this motion the moving system being no longer achromatic in itself, a relative displacement of the two coloured images takes place. Thus, the apparent separation of the objects, when the lens is not coaxial with the eye, differs from its true value, and settings will be in error. As it is not practically possible to ensure exact centering, with respect to the axis of the eye, of anything in the nature of a spectacle lens, the cure of chromatic parallax cannot be effected optically, and recourse must be had to devices such as have been described in § 8.

The author wishes to record his indebtedness to his colleagues Mr. T. Smith, Dr. J. S. Anderson, and his late colleague Mr. R. W. Cheshire for their interest in the problems with which the Paper deals ; and to Dr. Anderson, Mr. R. J. Trump, Mr. D. W. H. Bell and Miss A. B. Dale for their patience in taking long series of test observations on which some of the conclusions have been based.

ABSTRACT.

The Paper describes various parallax effects observable when objects illuminated by light of different colours are seen through a "pinhole" of smaller diameter than the pupil of the eye. The effects are due to the well-known chromatic aberration of the eye ; and it is shown that in most optical instruments of precision the necessary conditions for parallax of this nature are present when observations are made in the blue and violet ; and that this results in a serious diminution of the accuracy of such measurements. Several methods of obviating the difficulty are described.

DISCUSSION.

Prof. **Cheshire** said it was possible that the author had laid his finger on one of the main causes of the discrepancies which were found in many important determinations, those of refraction and dispersion of optical glass for example, between the results of different experimenters. He was struck by the expedient of using a fibre or wire at the slit, and was surprised that this method had not been employed before. Some interesting stereoscopic effects ought to be observed if two differently coloured objects were viewed through pinholes in front of each eye. If the pinholes were further apart than the centres of the pupils, the distance between the blue image in one eye and that in the other would be less than the distance between the red ones, and the blue object would appear further off. If the pinholes were closer than the centres of the pupils, the reverse would be the case, and the blue object would appear to be closer. In many opticians' windows the word FRIEND was exhibited, the letters F I and N being blue, while R, E and D were coloured red. On looking at this the red letters appeared to stand in front of the others. Prof. Von Rohr had once told him this was a true stereoscopic effect, the images on the two retinæ of a blue letter being closer together than the images of a red letter on account of the aberrations of the eye.

He had not given much thought to it himself, and would be interested if the author could explain the effect.

Mr. T. H. **Blakesley** said that if one formed a spectrum of a white light by a prism with refracting edge perpendicular to the slit, a drawn out image of the latter was obtained, which was white at the middle and coloured at the extremities. If this were now viewed through a pinhole which was moved to one side, he supposed the blue and red extremities would be deflected in opposite directions, while the central portion would be drawn out into a continuous spectrum. He did not think the refractive properties of the eye were sufficiently well known to justify actual numerical calculations of the Gaussian system.

Mr. S. D. **Chalmers** believed there had been a number of accurate determinations of the chromatic aberration of the eye. He was in the habit of assuming an aberration of ¾ Diopter (from C to F) in designing eyepieces, and for other purposes. He was interested in the author's wide slit method. He had himself found it better than using a fine slit to employ one of appreciable width and bisect it with the crosslines. Why not use a thick vertical wire in the eyepiece and a slit of three times the width. The setting obtained by bisecting the broad line with the wire was a very accurate one.

Mr. T. **Smith** pointed out that the result of § 5—viz., that the total parallax obtainable with a small fixed stop in front of the eye was independent of the positions of the objects—could be shown very simply. Owing to the stop the mean ray from any object was refracted at the same point of the refracting surface (approximately). The deviation of a blue ray incident at this point is independent of the angle of incidence, and $=\beta h$, where h is the distance from the axis, and $\beta=(\mu_b-1)/\mu_1 r$, r being the radius of the surface. Similarly, the deviation of a red ray incident at the same point is ρh, where $\rho=(\mu_r-1)/\mu_1 r$. The difference between the deviations of the two rays $=(\beta-\rho)h$, and is the parallax when the stop is at a distance h from the axis. Since β and ρ are independent of the inclinations of the incident rays, the parallax is independent of the distances of the objects from the eye. Mr. Guild's work in connection with these phenomena and their elimination in optical measurements had had important results. The precautions which he suggested had now been in operation at the laboratory for a considerable time, and the departmental records showed that the consistency of certain important determinations had been increased by at least five times. It was obviously essential to remove chromatic parallax if only because the errors due to it were so unsystematic.

Lieut.-Col. J. W. **Gifford** (communicated remarks) asked if the author had worked with an ordinary Ramsden eyepiece and shadow pointer—that is, a knife-edge pointer reversed so that there is no reflected light from the pointer. With shutters added so as to block out everything but the line under measurement he had found such an arrangement more accurate than any other he had used. Of course, the projected shadow of the point must be accurately focussed by the eyepiece for each line to be measured.

The **Author** replied as follows: I was not aware that many accurate determinations of the chromatic aberration of the eye had been made as stated by Mr. Chalmers. P. G. Nutting states in his Paper that the matter had rested with the first rough determinations of Helmholtz until his own experiments. I have not seen any work on this subject since then. Mr. Chalmers's statement that he makes an approximate allowance for this aberration in designing eye-pieces is of great interest. I have myself used a slit of appreciable width, bisecting it with the cross lines (inclined type), but do not like the arrangement nearly so well as the central fibre method described in the Paper; since, if the slit is sufficiently wide to render the cross clearly visible and free from diffraction troubles, the setting is not nearly so certain as that on a fine black line. With regard to the use of a thick vertical cross line and a slit of about three times its width, as suggested by Mr. Chalmers, this type of setting, in which the vertical wire is set so as to bisect the spectrum line, is hardly suitable for spectroscopic work for the following reasons: A symmetrical setting is certain to be estimated wrongly unless there is almost complete symmetry throughout the field of view. This is never the case in spectrum work unless there is only one line visible at a time. Further, the vertical wire is straight, whereas the spectrum lines are appreciably curved Thus the line can never be bisected for its whole length, and it is difficult to make a satisfactory bisection in the middle of the line when the condition does not hold for other parts of it. Any personal errors arising from this cause will come fully into the final result; for the zero reading to which the others are referred, being made on the unrefracted image of the slit, is not subject to this effect. A further objection, which applies to all wide slit methods, is the difficulty of close doublets. The sodium lines are involved in nearly all optical determinations, and it must be possible to deal with them under all degrees of dispersion from complete non-resolution to comparatively wide separation. The symmetrical setting of a vertical wire in a band of appreciable width is clearly bound to be influenced by the proximity of the adjacent bright band, even if the dispersion is sufficient to prevent overlap. If they overlap the setting is probably even worse. I have not seen the type of shadow pointer mentioned by Mr. Gifford, and so cannot speak of it from personal experience. From the description of it I conclude that it also is a wide slit method, and is therefore liable to some of the objections mentioned above when dealing with close lines. Further, the necessity of blocking out the rest of the field appears to be somewhat objectionable, on the ground that every extra adjustment which has to be made, especially near the end of the telescope or collimator, is a possible cause of error.

The spider line method advocated in section 8 is not, of course, a wide slit method in the sense of those mentioned above. Essentially, even the setting of the intersection of diagonal cross lines on a fine line, whether bright or black, is a symmetrical setting; but the area on which attention is concentrated is so much smaller than in the other types of setting considered that the disturbing effect of an adjacent line is practically inappreciable. In the foregoing it may seem as if the importance of small effects is overestimated; but it cannot be too strongly emphasised that the present state of optical science is such that any further researches on refraction and dispersion must, if they are to be valuable, attain an accuracy corresponding to about a second of arc in the angular measurements. This can never be attained if any possible source of error remains either in the design of the instrument or in the methods of using it. Prof. Cheshire's* suggestions are of considerable interest. The stereoscopic effect due to pin holes displaced in opposite directions in front of the two eyes can readily be observed A convenient arrangement is to mount the pin holes on the trial spectacles used by optometrists, when the distance between them can easily be adjusted to any desired value. If a number of rectangular slots cut in black paper and backed by alternate red and blue filters are used as objects, the effect is well marked, provided care is taken to see with both eyes, and not only with one.

I have not seen the actual optician's sign in which the word " Friend " in red and blue letters occurs. A hasty trial experiment, made before the meeting with the alternate red and blue slots mentioned above, convinced me that the red appeared to stand out from the blue quite as much when seen with only one eye as with both, and that the effect was monocular and not stereoscopic. This view appeared to be shared by Mr. Chalmers. Afterwards various observers were asked to come separately into the room in which the coloured slots were mounted, with one eye closed and with no knowledge of what they were to look for. In every case the red was said to appear appreciably nearer than the blue, and in no case was the effect increased by using both eyes. Thus there is undoubtedly a monocular effect, which is simply due to the fact that the images of the blue slots are formed in front of the red. The mind is accustomed to judge (from the necessary effort of accommodation) that if there are two images at different distances within the eye the anterior one is due to a more distant object, and so the blue slots are assumed to be further away than the red. Although no true stereoscopic effect was detected with these coloured slots, some further experiments appeared to show that there is a binocular effect in addition to the monocular one; and this arises from the fact that the line of sight is not coincident with the axis of the eye, but is inclined inwards with respect to it. The aperture of the incident beam, when the whole eye is used, is limited by the pupil, or rather by its virtual image in the refracting surface of the cornea; and the pupil is concentric with the axis.* The line of sight—which passes through the front nodal point—does not, therefore, pass through the centre of the pupil. The virtual pupil is roughly about 5 mm. anterior to the nodal point. The line of sight is inclined to the axis at about 6 deg., it therefore passes the virtual pupil at about 0·5 mm. from the centre, measured towards the nose. Thus the mean ray of the incident pencil, which passes through the centre of the pupil, is incident about 0·5 mm. to the outside of the line of sight; and is in consequence deflected inwards, a blue ray being more bent than a red. The difference in deviation from C to G' can be obtained by substituting 0·5 mm. for h in the formula for parallax in section 5; it is about two minutes, but will vary considerably in different eyes, since errors in the relative centering of the various parts of the eye amounting in some cases to 0·25 mm. are frequent. But, in general, blue images are formed slightly more to the nazal side of the retina than red ones (the objects being equally distant). Less convergence is therefore required to fuse the blue images when both eyes are employed, and so the blue object appears further away.

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* The reply to Mr. Cheshire is based on experiments performed afterwards.

The difference in deviation of blue and red rays coming from an object on the line of sight was experimentally verified by stretching a thin wire vertically in front of a red and blue filter, mounted one above the other and backed by illuminated ground glass. This was placed about three-quarters of a metre away, so that both halves of the wire were seen equally sharp. When looked at with one eye a distinct discontinuity of the wire at the junction of the filters was observed. If the right eye was used, the part of the wire on the blue field appeared displaced to the right; while with the left eye it appeared displaced to the left, each case corresponding to the greater nazal deviation of the blue rays. With both eyes open the stereoscopic effect was distinctly visible. These effects were also observed by persons with no foreknowledge of the possible results. It is clear that an error from this cause would result if one attempted to set red cross lines on a blue line, using the whole pupil of the eye, or to set black cross wires illuminated by red or white light on a blue line. No parallax would be observed, for movements of the eye would not alter the displacement of the mean ray from the line of sight; but a constant error would result

This source of error will affect observations made with low power instruments, in which the whole pupil is filled, if the cross lines are improperly illuminated. It is possible that this point has not been given sufficient attention in the design of low-power telescopes for use by night. Frequently the cross lines of such instruments are illuminated with a dull red light. The objects sighted are probably never so blue in colour as to introduce appreciable error from the cause discussed; but it would seem to be a safer practice to illuminate the cross lines with light from an intermediate part of the spectrum.

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* Usually; but it is often found displaced outwards from the axis, which would increase the effect discussed.

XXIV. *The Capacity of an Inverted Cone and the Distribution of its Charge.*

By Prof. G. W. O. Howe, *D.Sc.*

Received May 25, 1917.

The author has recently shown that the consideration of the equivalent capacity and inductance of a radio-telegraph aerial, consisting of a plain vertical wire, is simplified by assuming it to be an inverted cone.* In this connection it was considered desirable to calculate the ordinary electrostatic capacity of such a conductor, taking into account the effect of the earth with which the apex is almost in contact. So far as the author is aware this problem has not been previously solved.

The method employed is one which the author has already used on previous occasions for the calculation of the capacity of various types of aerials.† It consists essentially in considering the conductor as divided into a number of sections—the number depending on the accuracy required—each of which is subdivided into a large number of small elements, insulated from each other, and assuming that the charge is distributed uniformly over each section. Formulæ are then established for the average potential of each section, and the values of the densities on each section are so determined that this average potential is the same for each section, and, therefore, for the whole conductor. The effect of the earth is allowed for by the method of images. The only assumption made is that, when the charge has its natural non-uniform distribution, the potential does not differ appreciably from the calculated average value with the assumed artificial distribution of the charge. The accuracy of the method is limited solely by the number of sections into which the conductor is divided, and the consequent number of simultaneous equations which have to be solved.

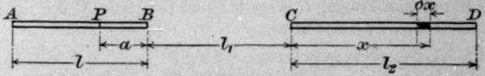

Fig. 1.

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* The Year-Book of Wireless Telegraphy, 1917, p. 694.

† Electrician, Vol. LXXIII., pp. 829, 859, 906; Vol. LXXV., p. 870; Vol. LXXVII., pp. 761, 880.

In the case of a cone it is advantageous to make a further simplification by replacing the conical sections by cylindrical ones of the same surface area, which, if the cone is of small angle, are approximately of the same mean diameter as the conical sections.

Since the average potential of any section is the sum of its average potentials due to its own charge and the charges on the other sections, it will be necessary to find the average potential of one conductor *AB* (Fig. 1) due to a uniformly distributed charge on another conductor *CD* in the same straight line.

If the conductor *CD* has unit charge per centimetre of length, the potential at *P* due to the element δx is $\delta x/(l_1+a+x)$, and that due to the whole conductor *CD* is

$$\int_0^{l_2} dx/(l_1+a+x)=\log\,(l_1+l_2+a)-\log\,(l_1+a)=V_p.$$

The average potential of *AB* due to the charge on *CD* is

$$\frac{1}{l}\int_0^l V_p da=\frac{1}{l}\Big\{(l+l_1+l_2)\log\,(l+l_1+l_2)-(l_1+l_2)\log\,(l_1+l_2)$$
$$-(l+l_1)\log\,(l+l_1)+l_1\log l_1\Big\}.$$

If $AB=CD=l$, this becomes

$$\frac{1}{l}\Big\{(2l+l_1)\log\,(2l+l_1)-2(l+l_1)\log\,(l+l_1)+l_1\log l_1\Big\}.$$

In our case the gap l_1 is made equal to $o, l, 2l, 3l$, &c., which leads to further simplification.

If $l_1=o$, the average potential $=2\log 2 \qquad =1{\cdot}3863$
$l_1=l$,, ,, ,, $=3\log 3-4\log 2=0{\cdot}52323$
$l_1=2l$,, ,, ,, $=4\log 4-6\log 3$
$+2\log 2=0{\cdot}3398$
$l_1=3l$,, ,, ,, $=5\log 5-8\log 4$
$+3\log 3=0{\cdot}25271$
$l_1=4l$,, ,, ,, $=6\log 6-10\log 5$
$+4\log 4=0{\cdot}20132$

. . . .

$l_1=nl$,, ,, ,, $=(n+2)\log\,(n+2)$
$-2(n+1)\log\,(n+1)+n\log n.$

The average potential of a straight wire due to its own uniformly distributed charge has previously been shown to be equal to $2\log\,(l/r)-0{\cdot}6137$ for unit charge per centimetre.*

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* Electrician, Vol. LXXIII., p. 830, 1914.

These formulæ can now be applied to the case of the inverted cone.

In order to illustrate the effect of the degree of subdivision, it will be assumed in the first place that the whole cone is replaced by a cylinder of the same mean radius; the cone will then be subdivided into two sections, then into three and finally into four.

1. The whole cone replaced by a single cylindrical section (Fig. 2). Let h=height of the cone, r=mean radius of the cone. Assume a uniform charge of 1 unit per centimetre of length, then the average potential due to its own charge is $2 \log (h/r) - 0{\cdot}6137$, whilst that due to the equal and opposite charge on the image is $-1{\cdot}3863$. Hence, the resultant potential is $2 \log (h/r) - 2$ and the capacity

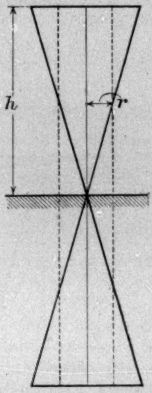

FIG. 2.

$$= \frac{h}{2 \log (h/r) - 2} \text{ cm.}$$

If h=100 ft and r=2 mm. the capacity is 176·6 cm.

2. The cone divided into two sections, each being replaced by a cylinder of the same mean radius (Fig. 3)—

For the upper section, $l = h/2$, radius$=3r/2$, charge$=q$ units per centimetre of length.

For the lower section, $l = h/2$, radius$=r/2$, charge$=1$ unit per centimetre of length.

The average potential of the upper part due to—

its own charge $=q(2\log(h/3r)-0{\cdot}6137)$,
the lower part $=1{\cdot}3863$,
the image of the lower part $=-0{\cdot}52323$,
the image of the upper part $=-0{\cdot}3398\,q$,

giving a resultant average potential of—

$$(2\log(h/3)r-0{\cdot}9535)q+0{\cdot}8631.$$

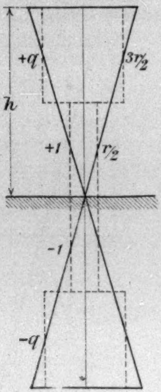

FIG. 3.

The average potential of the lower part due to—

its own charge $=2\log(h/r)-0{\cdot}6137$,
the upper part $=1{\cdot}3863\,q$,
the image of the lower part $=-1{\cdot}3863$,
,, ,, upper part $=-0{\cdot}52323\,q$,

giving a resultant average potential of

$$0{\cdot}86307q+2\log(h/r)-2.$$

On equating the average potentials of the two sections it is found that

$$q=\frac{2\log(h/r)-2{\cdot}8631}{2\log(h/r)-4{\cdot}01379}.$$

Hence, the upper section has the greater charge per unit length.

If, as before, $h=100$ ft. and $r=2$ mm.,

$$q=1{\cdot}075,$$

the average potential is 18·193, the total charge 3,163 units, and the capacity 174 cm.

3. The cone divided into three sections, each replaced by a cylinder of the same mean radius. The length of each section is $h/3$, their radii are $r/3$, r and $5r/3$, and the charges per centimetre are 1, q, and q_1 respectively.

The average potential of the upper section due to—

its own charge and image $=q_1(2\log(h/5r)-0{\cdot}6137-0{\cdot}20132)$,

the mid-section and its image $=q(1{\cdot}3863-0{\cdot}25271)$

the apex section and its image $=0{\cdot}52323-0{\cdot}3398$,

giving a resultant average potential of

$$q_1\,(2\log\,(h/5\,r)-0.81502)+1.13359\;q+0.18343.$$

Similarly, it is found that the average potential of the middle section is

$$q(2\log(h/3r)-0{\cdot}9535)+1{\cdot}13359q_1+0{\cdot}86307,$$

whilst that of the apex section is

$$2\log(h/r)-2+0{\cdot}86307q+0{\cdot}18343q_1.$$

On equating these three values of the average potential and substituting the values of h and r already assumed, it is found that

$q=1{\cdot}0055$, $q_1=1{\cdot}1171$, the average potential $=18{\cdot}3376$, the total charge 3,174 units and the capacity 173·2 cm.

4. The cone divided into four sections each replaced by a cylinder of the same mean radius. Except that they are more laborious, the calculations in this case are similar to those for the case just considered. If $h=100$ ft. and $r=2$ mm. the results obtained are as follows :—

Charge per centimetre of length on the apex section = 1.
,, ,, ,, ,, next ,, = 0·9936.
,, ,, ,, ,, next ,, = 1·031[illegible].
,, ,, ,, ,, top ,, = 1·1458.
Total charge = 3,180 units.
Average potential = 18·41.
Capacity = 172·75 cm.

The results are plotted in Fig. 4, from which it is seen that the capacity of the cone is about 171·5 cm. In the upper part of Fig. 4 the charge per centimetre of length in the case of the subdivision into four sections is plotted, and a curve drawn through the mid-point of each section. This shows the approximate distribution of the charge over the actual cone. It will be seen that over the lower half of the cone the charge has practically a constant value per unit length, whilst above the mid-point the charge per unit length increases rapidly, and at the upper extremity is almost proportional to the radius; in other words, the density over the lower half is inversely proportional to the distance from the ground, but above the mid-point it falls off much less rapidly, and is almost uniform near the upper extremity.

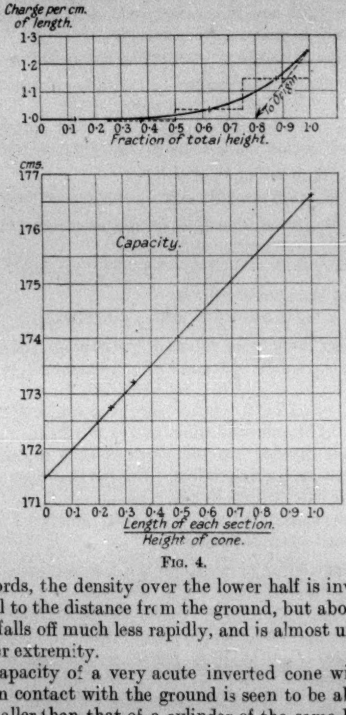

Fig. 4.

The capacity of a very acute inverted cone with its apex almost in contact with the ground is seen to be about 2·9 per cent. smaller than that of a cylinder of the same height, with a radius equal to the mean radius of the cone. This can be calculated by the formula

$$C=\frac{h}{2\log(h/r)-2}\,cm.$$

XXV. *On the Measurement of Small Inductances and on Power Losses in Condensers.* By Albert Campbell, *B.A.* (*From the National Physical Laboratory.*)

Received June 7, 1917.

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Small Inductances : Methods 1 and 2.
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Frequency (Sifter Method).
Sifter Method with damped Oscillations.

Introductory.

The measurement of very small self inductances of circuits having only two terminals is a matter of no great difficulty, if the resistances are not high.

One of the simplest ways of doing this is by Heaviside's method,[*] in which comparison is made with a known mutual inductance.

It is easy to build a mutual inductometer of almost any desired lowness of range by using the device of stranding the windings of the coils as I have described in an earlier Paper. The lowest range of the instrument used in the measurements here described was from 0 to 1 microhenry, the scale being readable to 0·001 microhenry at the upper part. An instrument of this kind combined with a constant inductance rheostat (and ratio arm coils) is sufficient for dealing with two-terminal resistances. This bridge system unfortunately is not directly applicable to four-terminal resistances, in which the potential terminals are distinct from the other two. Venner [†] has shown, however, that the inductances of such resistances can be determined by the addition of the Kelvin double bridge device to the Heaviside bridge.

Other methods not of bridge type have been used. Orlich [‡] employed an electrostatic wattmeter, and his method was used later by E. and W. H. Wilson.*

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[*] Phil. Mag., p. 173, Vol. 23, Feb., 1887. When I described and elaborated this method (Proc. Phys. Soc., p. 69, Vol. 21, 1908) I was not aware that it was one of Heaviside's very extensive series.

[†] Bulletin of the Bureau of Standards, p. 559, Vol. 8, 1912.

[‡] Zeitschrift für Instrumentkunde, p. 114, Vol. 25, 1905.

More recently several ingenious methods, chiefly using two-phase currents, have been described. In Silsbee's Paper† there is an interesting discussion of "impure" mutual inductances, namely, those in which there is internal power loss (by eddy currents or dielectric hysteresis, &c.), and by a useful convention the time constant of such circuits is defined.

Methods Using M, R Element.

The methods which I proceed to describe are simple in their working, requiring only a single-phase source of current; on the other hand, they have certain limitations in practice, which will be discussed below. They are developments of the method of testing transformers which I described some years ago.‡ The fundamental principle in that method was the introduction of the elementary pair (M, R) shown in Fig. 1,

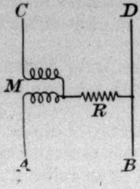

FIG. 1.

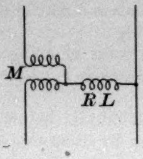

FIG. 2.

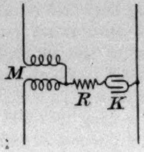

FIG. 3.

where *A B* are current terminals and *C D* potential points (or vice versa).

Other methods of utilizing this element have been developed by other observers,§ and it has formed the basis of the simplest alternating-current potentiometer, that of Prof. A. Larsen,‖ of Copenhagen.

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* Electrician, p. 464, Vol. 56, June, 1906.

† Silsbee, Bulletin of the Bureau of Standards, p. 375, Vol. 13, 1916.

‡ Report of National Physical Laboratory, March, 1909, and Proc. Phys. Soc., p. 497, Vol. 22, 1910.

§ C. H. Sharp (in Discussion, June 30, 1909), Trans. American Inst. of Elec. Engs., p. 1040, Vol. 28 (2), 1910; Sharp and Crawford, Trans. A.I.E.E., p. 1517, Vol. 29 (2), 1911; Agnew & Silsbee, Trans. A.I.E.E., p. 1635, Vol. 31, 1912.

‖ ("Der Komplexe Kompensator") Electrotechnische Zeitschrift, p. 1039, Vol. 41, Oct., 1910; and Electrician, p. 738, Vol. 66, Feb. 17, 1911 A mistranslation ("Compensation apparatus" for "potentiometer") in the title obscured the meaning for most English readers.

The elementary pair (M, R) may be further developed by associating with *R* either self-inductance or capacity as in Figs. 2 and 3. These two cases require the same mathematical treatment, but they have different applications in practice.

I shall first discuss the case of self-inductance.

Method 1.—In Fig. 4 let *r* be a resistance with potential terminals, having self-inductance *l* to be found. It is connected, as shown, with a low-reading inductometer giving mutual inductance *m*, and a pair of inductive coils whose mutual inductance *M* can be varied. The second of these coils (of resistance *R* and self-inductance *L*) is connected to a resistance *S* having small self-inductance *λ*. (*S* may be adjustable by a slide-wire, in which case *R* and *L* must include the part above the slider.) *A* is a source of alternating current and *G* a vibration galvanometer or telephone.

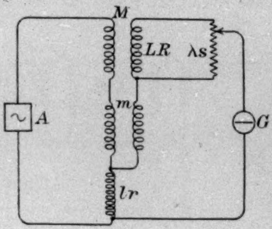

FIG. 4.

If $\omega=2\pi n$, where n is the frequency of the source, it is easy to show that

$$MS=r(L+l)+(R+S)(\ +m), \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and

$$(L+\lambda)(l+m)\,\omega^2-\lambda M\,\omega^2=r(R+S). \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now, let L/R be (relatively) large, and λ/S small and λM negligible compared with $L\,(l+m)$; then equation (2) becomes

$$L(l+m)\,\omega^2=r(R+S), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

which gives *l* conveniently without requiring an exact knowledge of *M* and *λ*.

In practice a disadvantage of this method is that usually the proper conditions are not easily obtained unless the frequency is tolerably high (say, 800 ∼ per second). It should be noted, however, that, as Bêthenod * and Orlich † have shown, the resistance and self-inductance of a properly designed shunt remain practically constant up to such frequencies.

Example.—The shunt to be tested had a resistance $r=0{\cdot}01$ ohm. With $R+S=5{\cdot}99$ ohms, $S=0{\cdot}274$ ohm, $L=0{\cdot}0100$ henry, λ less than 1 μ henry, $M\doteqdot 370$ μ henries, a balance was obtained when m was $0{\cdot}177$ μ henry, the frequency n being 800 ∼ per second. Hence, from equation (3), $l=0{\cdot}063$ μ henry. The current used was of the order of 0·1 ampere, and was obtained from a small buzzer.

Method 2.—A method more generally applicable is shown in Fig. 5, r being the resistance whose self-inductance l is to be

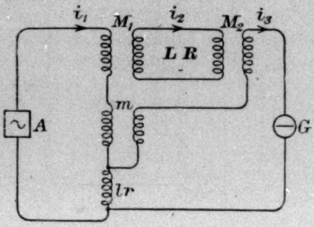

FIG. 5.

determined, and m a low-reading inductometer. The primary circuit is also linked to the galvanometer circuit by an intermediary closed circuit having resistance R and self-inductance L. The linking mutual inductances should be variable, but in general need not be known. There should be no direct mutual inductance between the primary circuit and the galvanometer circuit other than m. This condition can always be checked by opening the intermediary (L, R) circuit; then, when r is cut out m should read zero. The best plan is to put the four coils forming M_1 and M_2 at a good distance from the m inductometer, and to turn them so as to be conjugate to each other pair and to the inductometer coils.

Let the instantaneous values of the currents in the three circuits be i_1, i_2 and i_3 as shown; and let $a=\omega\sqrt{-1}$.

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* Jahrbuch der drahtlosen Tel., p. 397, Vol 2.

† Zeitschrift für Instrumentenkunde, p. 241, Vol. 29, 1909.

Then, when $i_3=0$, we have

$$(R+La)i_2+M_1ai_1=0,$$

and

$$[r+(l+m)a]i_1+M_2ai_2=0,$$

and hence

$$R(l+m)+rL=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

and

$$Rr=(l+m)L\omega^2-M_1M_2\omega^2. \quad . \quad . \quad . \quad . \quad . \quad (5)$$

As mutual inductances are reversible we may take their values positive or negative in general Changing the signs of M_1 and m (and with $m>l$), we have

$$\frac{m-l}{r}=\frac{L}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and

$$Rr=[M_1M_2-(m-l)L]\omega^2. \quad . \quad . \quad . \quad . \quad (7)$$

Equation (6) gives l without a knowledge of the values of M_1 and M_2.

From (6) and (7) we have

$$r=\frac{RM_1M_2\omega^2}{R^2+L^2\omega^2}\doteq\frac{M_1M_2\omega^2}{R},$$

when $L\omega/R$ is small,

and

$$m-l=\frac{LM_1M_2\omega^2}{R^2+L^2\omega^2}\doteq\frac{LN_1M_2\omega^2}{R^2},$$

where $L\omega/R$ is small.

The following two examples will give an idea of the relative magnitudes of the quantities involved for two different frequencies :—

r ohm.	l henry.	n ∾ per sec.	m/l.	R ohms.	micro-henries. L	M_1	M_2
0·01	0·05	100	2	10	50	250	1000
0·01	0·05	800	2	50	250	200	100

From equation (6) we see that $(m-l)$ must always be positive. From (6) and (7) we have

$$Rr=[M_1M_2-L^2r/R]\omega^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence, M_1 M_1 must always be greater than L^2r/R.

Current Transformer Compensated for Frequency.

The elementary pair (M, R) of Fig. 1 is of interest in connection with current transformers. In an ironless transformer let M be the mutual inductance, and let Q and L be the resistance and self-inductance of the secondary circuit. Then, If I_1 and I_2 are the effective values of the primary and secondary currents, it is well known that

$$\frac{I_1^2}{I_2^2}=\frac{Q^2+L^2\omega^2}{M^2\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (9)$$

and accordingly the current transformation ratio is not constant for different frequencies.

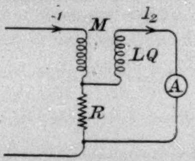

Fig. 6.

Now let a resistance R be inserted as shown in Fig. 6. Then we have

$$\frac{I_1^2}{I_2^2}=\frac{(R+Q)^2+L^2\omega^2}{R^2+M^2\omega^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (10)$$

The ratio is independent of ω and equal to L/M, if

$$(R+Q)/R=L/M. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (11)$$

When this condition is satisfied the ratio of current transformation will be constant for all frequencies, and hence also for all wave forms.

Determination of Capacity in Terms of Mutual Inductance and Frequency.

In a former communication* I described a very simple method of measuring capacity in terms of mutual inductance and frequency. The method has proved very useful for quick and accurate determinations of frequency by the help of a known condenser and a variable mutual inductance.

With the method in its simplest form, as shown in Fig. 7, the mutual inductance m is adjusted until G, the vibration galvanometer (or telephone) shows no current. Then, $mk\omega^2=1$.

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* Proc. Phys. Soc., p. 69, Vol. 21; and Phil. Mag., p. 155, Jan., 1908.

But unless the condenser is free from leakage (or other power loss), a perfect balance cannot be obtained, but only a minimum current in the detecting instrument. With an imperfect condenser, however, this trouble can be entirely eliminated by adding an auxiliary closed circuit similar to that

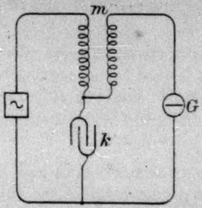

FIG. 7.

in Fig. 5. By this device the condenser losses can always be compensated for and a perfect balance obtained. The complete method is shown in Fig. 8, in which the imperfection of the condenser is represented by the series resistance *r*.

Let the current in *G* be reduced to zero by adjusting *m* and M_1 *or* M_2.

By writing $-1/k\omega^2$ for l in equations (6) and (7) and changing the sign of *m* the conditions for balance are obtained, namely

$$1/k\omega^2=m+rL/R, \quad \text{. . . . .} \quad (12)$$

and

$$Rr=[M_1M_2+(m-1/k\omega^2)L]\omega^2 \quad \text{. . . .} \quad (13)$$

Hence

$$Rr=[M_1M_2-L^2r/R]\omega^2. \quad \text{. . . . . .} \quad (14)$$

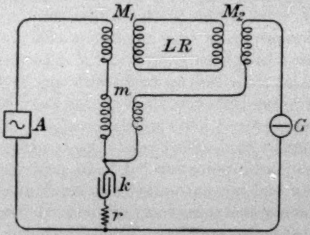

FIG. 8.

Here also M_1M_2 must be greater than L^2r/R, except in the limiting case of a perfect condenser, where $r=0$ and then M_1M_2 also $=0$.

From the above equations we have

$$r=RM_1M_2\omega^2/(R^2+L^2\omega^2), \quad . \quad . \quad . \quad . \quad (15)$$

and

$$1/k\omega^2=m+LM_1M_2\omega^2/(R^2+L^2\omega^2) \quad . \quad . \quad (16)$$

When ω is known, these equations give k and r and hence $rk\omega$, the power factor of the condenser.

When the method is used for measuring frequency, it is best to arrange the auxiliary closed circuit so that L/R is very small, and hence rL/R negligible compared with m.

Then

$$\omega^2=1/km, \quad . \quad . \quad . \quad . \quad . \quad ; \quad (17)$$

and the frequency is determined as though the condenser were perfect, and without a knowledge of the values of r, L, R, M_1 and M_2, the auxiliary circuit merely introducing a small vector which balances the ri_1 without perceptible effect in equation (17).

In general the auxiliary closed circuit can be a few turns of wire mounted so as to be movable close to the coils of the inductometer giving m. Sometimes even a coin shifted about above the coils is sufficient to make the balance exact. With a given condenser the scale of the inductometer can be graduated to read the frequency directly, and with a suitable series of condensers the same scale may be used with even multipliers.

When the wave form of the source of current contains strong harmonics these may somewhat obscure the point of balance when a telephone is used as the detecting instrument. For frequencies from 200 up to 5,000 ∽ per second, the harmonics cause very little trouble if the primary self-inductance in the inductometer is kept relatively high. [The system of Fig. 4 can also be used with a condenser in place of l.]

In a former communication * I have described the use of the condenser and mutual inductance combination as a wave sifter by which a component of any desired frequency can be suppressed in a given circuit. The addition of the auxiliary closed circuit here described makes the sifting perfect even when quite common paraffin paper condensers are used.

When the power loss in the condenser (and hence its power factor) is to be determined, usually it is best to keep L/R very small.

Then

$$r=M_1M_2\omega^2/R. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

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* Proc. Phys. Soc., p. 107, Vol. 24, 1912.

In this case L need not be known accurately, but M_1M_2 must be determined. The same care must be taken with regard to the position of the coils forming M_1 and M_2 as in Method 2.

Or M_1M_2 may be directly determined by adding a known resistance s to the condenser circuit and altering R to R' to obtain a rebalance.

Then $$M_1M_2\omega^2=s/(1/R'-1/R). \quad . \quad . \quad . \quad . \quad . \quad (19)$$

In this case the loop (L, R) may be placed over the inductometer coils if desired.

The following example gives an idea of the relative values of the various inductances and resistances in the determination of the power factor of a good quality mica condenser of capacitance 0·1 μF and power factor 0·0005 using a frequency of 800 ∾ per second.

Here $\omega=5000$, $m=0{\cdot}4$ henry, and $r=1$ ohm.

If $R=10$ ohms, then $M_1M_2=4\times10^{-6}$, and M_1 and M_2 can be 2 millihenries each.

It should be remarked, however, that, altho' this method of determining the power factor of a condenser may be a good one for certain cases, for general use the Carey Foster method appears to hold the highest place for simplicity and accuracy.

Sifter Method with Damped Oscillations.

If the source gives damped oscillations of the form $\varepsilon^{-bt}\cos\omega t$, where $b=n\times$ log. decrement, then (Fig. 7) it is easy to show that, for a balance,

$$mk(\omega^2+b^2)=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

and $$2mb=r, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where r is the series resistance including that which represents the loss in the condenser. Here no auxiliary circuit is required to balance the loss, if by external addition r can be made to satisfy equation (21).

In conclusion, I would express my thanks to Sir Richard Glazebrook, F.R.S., for kind interest in the work.

XXVI.—*The Mechanism of Colour Vision.* *By* J. GUILD, *A.R.C.S., D.I.C., F.R.A.S.*

RECEIVED JULY 2, 1917.

IN a recent Paper* to the Royal Society, Dr. R. A. Houstoun has outlined a new theory of the mechanism of colour vision, in which the three sets of resonators of the Young-Helmholtz theory are replaced by a single set, of which the resonance point corresponds to $\lambda=0{\cdot}55\mu$.

The present Paper is primarily a criticism of the suggested theory and of the premises from which many of its main features are deduced.

The theory supposes the existence in the eye of a great number of vibrators with a free period in the green, and it is assumed that the motion of one of these may be represented by the equation

$$\frac{d^2x}{dt^2}+h\frac{dx}{dt}+n^2x=E\cos\omega t, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

x being the displacement from the position of rest at time t, and $E\cos\omega t$ the force per unit mass exerted on it by the incident light wave. Starting from this, it is deduced that the ratio of the absorbed to the incident energy of any wave-length, λ, is proportional to the quantity,

$$\frac{\lambda^2}{(\lambda^2-a^2)^2+b^2\lambda^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where a is the resonance wave-length and b is another constant. It is then further assumed that the luminosity of the light is simply proportional to the energy absorbed by the resonators, and that the above expression, therefore, represents also the ratio of the luminosity to the incident energy, *i.e.*, the visibility of the radiation of wave-length, λ.

Suitable values of a and b being chosen, the expression is plotted in comparison with the experimental visibility curve.

There are several assumptions underlying this treatment which do not appear to the present writer to be justified. In the first place, without more knowledge than we at present possess concerning the nature and properties of the resonators in question, the most general significance must be given to the quantities occurring in the oscillation equations ; and to adopt values for the constants a and b which give the closest agreement with an experimental curve seems a somewhat arbitrary proceeding. Moreover, it is quite possible—if it is not, indeed, probable—that the behaviour of the oscillators cannot be represented by such a simple expression; h or n, or both, may not remain constant over the range of displacements and rates of change of displacement occurring in vision ; in which case neither a nor b in the above expression for the ratio of absorbed to incident energy can be regarded as constant. It might easily be that the oscillations depart so far from these conditions that the form of the expression obtained for the absorbed energy is quite misleading.

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* "A Theory of Colour Vision," Proc. Royal Soc., Vol. 92, p. 424.

The second assumption, that the luminosity is proportional to the amount of energy absorbed by the resonating system, *whatever the wave-length*, appears not only to be devoid of justification, but to be very improbable. It has to be borne in mind that we are not here comparing physical effects with their appropriate physical causes : luminosity is a measure of the *sensation* produced—*i.e.*, it is a mental, not a physical, phenomenon ; and in passing from the physical processes occurring at the retina to the sensations to which they give rise in the brain we have at some stage to cross the unknown bridge between matter and thought. Knowing, as we do, nothing whatever of the process whereby energy, functioning on the physical plane as vibrations of a sense organ, is made to function on the mental plane as the corresponding sensation, it is obviously impossible to take for granted, as Dr. Houstoun does, that the conversion factor, if we may employ the term, from stimulus to sensation is independent of the frequency of the former. Even before this stage is reached, the energy absorbed by the resonators has to pass to the brain through a complex system of tissue layers and nerve fibrils. To assume without evidence that impulses of all frequencies are transmitted with equal efficiency by such a system can hardly be justified.

Thus, from the considerations outlined above, it would appear that :—

(*a*) There is some reason for doubting that the expression obtained by Dr. Houstoun from the properties of a simple type of resonator really represents the ratio of absorbed to incident energy for wave-length λ in the case of the eye.

(*b*) Even were this ratio known with certainty, there is no justification for regarding it as a measure of the relative visibility of radiation of different wave-lengths.

The agreement of Dr. Houstoun's curve with the experimental curve strikes the writer as very superficial, and seems to prove nothing beyond the fact that any mathematical curve which has a peak in it will, if plotted to a suitable scale, with its constants suitably chosen, bear a considerable resemblance to any other peaked curve.

Fechner's Law.

This law, which is sometimes referred to as Weber's, states that the smallest increment of intensity which the eye can detect bears a constant ratio to the total intensity.

In order to make his expression for visibility fit in with this law, Dr. Houstoun evolves a scheme whereby an ever-increasing number of resonators is thrown out of action as the intensity of the incident light increases.

This appears to be wholly arbitrary and unnecessary. The established fact is that $dI/I=$constant, where dI is the minimum difference in intensity which the eye can detect when the total intensity is I. Dr. Houstoun then assumes that $dS \propto dI/I$, where dS is the increment of sensation produced by an increment, dI, in the incident intensity. But the increment of energy absorbed by *each* resonator is proportional to dI, since expression (2) is constant for any given wave-length, and this necessitates a progressive diminution in the number of active resonators, as the intensity increases, in order to account for the lower rate of increase of the sensation. Reverting to the relation $dS \propto dI/I$, it follows from the fact that the smallest detectable value of dI/I is constant that the smallest detectable value of dS is also constant—*i.e.*, the relation $dS \propto dI/I$, with all the consequences which it involves, is based on the assumption that the smallest increment in the *sensation* of light which the mind will notice is constant, whatever the magnitude of the sensation to which it is added. As well might we expect that because the addition of an extra inch to a foot rule would be detected readily, the addition of an inch to a mile would also be noticeable.

This is only another example of the danger already mentioned of leaving out of account the unknown process by which physical causes produce mental effects, and of assuming that all peculiarities in sensation must necessarily be due to peculiarities of the sense-organ concerned.

In the view of the writer, Fechner's law is psychological, and not physiological, in origin ; it is merely a particular case of a general law which applies to all five senses, and which may be expressed by saying that the mind takes cognisance of *relative* magnitudes only. The application of this law to some of the other senses is a matter of everyday observation ; and is so very well known that nobody troubles to notice it, still less to represent it in terms of differentials or logarithms. For example, we have no difficulty in telling which of two weights is the heavier where one weighs half an ounce and the other a quarter, but it is a very different matter if one is fifty pounds and the other fifty pounds and a quarter of an ounce. As a matter of fact, the minimum difference which we can detect between two weights from their feeling of heaviness is very roughly proportional to their magnitude, yet no one would suggest that, with increasing load, there is a progressive diminution in the number of active nerve centres in the muscles of the hand or arm to account for our inability to detect the additional weight when a quarter of an ounce is added to fifty pounds.

Many similar examples could be quoted in connection with other senses. For instance, a few grains of common salt dissolved in a glass of water is obvious to the sense of taste, but the same few grains added to water already moderately saline would pass unnoticed.

In a quiet room the ticking of a clock or the slightest whisper is distinctly audible, but in a busy street or workshop we may have to shout in order to be heard. Yet we do not find that part of the mechanism of the ear goes out of action as the volume of sound increases. In fact, we know that, very approximately, the disturbances of the tympanum (and other parts of the auditory apparatus), due to different sounds, are superposed, and that the amount of energy absorbed by the ear from a particular sound is approximately independent of whether it is the only sound heard or one of many. Nevertheless, whether we are conscious of hearing it or not depends on its relative importance to the total amount of sound-sensation which the mind is perceiving at the moment.

Examples of this kind could be multiplied almost indefinitely, showing that whenever the mind deals with magnitude it is *relative* magnitude only with which it concerns itself ; and a given increment in any magnitude will be perceived only if it produces a sufficiently important alteration in the total magnitude.

Enough has probably been said, however, to make it clear that Fechner's law has not to be explained by retinal peculiarities, as attempted by Dr. Houstoun; and that it does not, as he supposes, involve (for a given wave-length) the non-proportionality of sensation to intensity of incident light. On the contrary, the writer believes that there is at least presumptive evidence for regarding dS/S=constant as a fundamental psychological law connecting the magnitude of a *sensation* with the smallest change in that *sensation* which the mind will detect; and that, therefore, where we find a similar law—such as Fechner's—holding between the minimum detectable change in the *external impulse*, and the total magnitude of that *impulse*, such external law proves, within the limits in which and to the order of accuracy with which it holds, that the sensation is proportional to the physical impulse producing it. Thus Fechner's law simply shows that dI/I=constant=dS/S; and that, consequently, S is proportional to I. There is, therefore, no necessity for assuming a diminution in the number of active resonators at high intensities.

According to this view, it is not Fechner's law that requires a physiological explanation, but rather the fact that it only holds approximately; but when one considers the nature and complexity of the mechanism by which we rob the light wave of its energy and convey it to the brain it is surprising that Fechner's law holds even as well as it does.

After disposing of Fechner's law by ascribing to his resonators the very arbitrary properties already mentioned, Dr. Houstoun discusses the Purkinje effect. Rejecting the hypothesis that this is due to the progressive change from rod to cone vision as the intensity increases, he advances two possible explanations to explain the effect with one system of vibrators only.

The first of these assumes the vibrators to be embedded in a yellow medium, and would, therefore, confine the phenomenon to the region of the macula lutea; since, outside this small patch, no such yellow colouring matter exists. This is contrary to experience, since it is outside the yellow spot that the effect is most marked. In fact, in the centre of the fovea, where there is plenty of yellow colouring (*but no rods, and therefore no possible change from rod to cone vision*), the Purkinje effect is said to be absent altogether.

The second explanation—to which the author of the theory himself inclines—assumes that the resonators are not all of one frequency, but have frequencies grouped about a mean. There is nothing inherently improbable in this assumption; but the explanation also assumes the diminution, at higher intensities, of the effective number of resonators, an assumption which has already been criticised as being based on an erroneous conception of the origin of Fechner's law.

In rejecting the usual explanation of the Purkinje effect, Dr. Houstoun seems to leave out of account many undisputed facts connected with it. He does not think it necessary to "assume" two different mechanisms; but the existence of the rods and cones, and the differences between them, are not matters of assumption, but well-known physiological facts. Further, the fact that the sensation of light persists in those portions of the retina where there are plenty of rods, down to far lower intensities than within the fovea, where there are no rods, if it does not absolutely prove the ordinary view that the rods are chiefly responsible for vision at low and the cones at high intensities, at least makes it a much more likely and legitimate assumption than any of those with which Dr. Houstoun attempts to replace it.

There is still another point at which the theory joins issue with experimental fact. We may profitably quote from the original: "But, it will be asked, how does this theory explain the apparent trichromatism of our ordinary sensations? We must here fall back on the reason give by Dr. Edridge-Green—namely, that the colour-perceiving centre in the brain is not sufficiently developed to discriminate between the character of adjacent curves. Two curves must be widely different in shape and position before the colour-perceiving centre can detect the difference. A curve has an infinite number of points on it; the colour-perceiving centre is so badly developed that, as far as it is concerned, the curve is sufficiently specified by three points on it, provided that these points are distributed over the spectrum. We can, therefore, represent our energy curve by three points."

If this view were correct, a given colour could be matched sufficiently well to satisfy our "badly-developed" colour perception with suitable proportions of any three colours so long as these were "distributed over the spectrum." Certainly it should be possible to select the red, green or violet constituents from within a fairly wide range.

Experiment has abundantly shown, however, that unless the three constituent colours are chosen of wave-lengths lying within very narrow limits our colour perception is so *well* developed that it is impossible to reproduce other colours to our satisfaction. The three primary wave-lengths of the trichromatic theory cannot, therefore, be regarded as three points on a single curve with no further significance than that they are reasonably spaced over the spectrum. Experiment has fixed them for us within much narrower limits than is consistent with this view.

Dr. Houstoun claims the fact that the colour of monochromatic light becomes whiter at high intensities as supporting his theory; but this phenomenon is also accounted for with perfect ease on the trichromatic theory, and cannot, therefore, be cited as a discriminant.

In conclusion, it may be pointed out that the proposed theory, even if the validity of all the assumptions which the writer has criticised be allowed, does not do more than explain in an approximate manner the general characteristics of *normal* colour vision. The author makes no attempt to account for the multitude of phenomena of complete and partial colour blindness which have served to establish the trichromatic theory both qualitatively and quantitatively. The only reference to colour blindness is contained in a more popular résumé of the theory given in Science Progress,* in which the author says: "Instead of accounting for colour blindness by a want of discriminating power in the colour-perceiving centre of the brain, we may ascribe it to excessive disturbance of the vibrators in the retina. This would make the energy curves widen out and lose their distinctive form." It is scarcely necessary to point out that colour blindness is not accounted for on the tri-chromatic theory by want of discriminating power in the brain; but is supposed to be due to one of the three sets of resonators being wholly or partially out of action. The difficulty of a red-blind man with a deep red light is not that he cannot tell its colour, but that he cannot see it. How excessive disturbance of the vibrators should result in absence of sensation in such cases is difficult to follow. On this theory the brightness of a white light should be greater in the case of a man who is wholly or partially colour blind on account of this excessive disturbance of the vibrators; whereas, as is definitely established, it is less.

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* No. 43, January, 1917, p. 387.

After all, the objection which Dr. Houstoun has to the trichromatic theory appears to be summed up in this: There is no physiological evidence of the existence of three sets of vibrators. Why should there be? It is very improbable that the rods and cones are themselves the resonators in question. It is much more likely that the resonators consist of specialised cells of ultra-microscopic—possibly molecular—dimensions within the substance of the rods and cones; in which case there is no physiological evidence of the existence of any of them. As a matter of fact, no one would associate anything in the microscopic structure of the retina with resonating systems at all. The evidence for the existence ot any such system is wholly indirect: they have to be assumed to explain the known physical properties of the eye. If these properties include, as they must, the accumulated facts of colour blindness, it is difficult to see how the assumption of three sets of resonators is to be avoided. There may be no direct evidence of their existence; but there is plenty of evidence of the necessity for their existence.

XXVII.—*The Use of Monochromatic Interference Rings for the Measurement of Curvature.* *By* S. D. CHALMERS, *M.A., Technical Optics Department, Northampton Polytechnic Institute.*

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SHALLOW convex surfaces can be compared directly with a plane surface. A measuring microscope is provided with a vertical illuminator arranged to direct monochromatic light on the stage; the convex surface is placed on the plane and the whole carefully centred. If necessary, the edges of the lens are supported to prevent rocking. The rings are focussed and carefully centred, and the diameters of convenient rings measured, say, the 3rd and 20th rings. Let the measured diameters be $2C_1$ and $2C_2$; then the radius of curvature of the plate is given by

$$r_1=\frac{C_2^2-C_1^2}{(20-3)\lambda}.$$

The difference $20-3=17$ is chosen because $17\times0{\cdot}0005894$ is approximately 0·01 mm., and the value of the radius is readily calculated as accurately as the measurements permit.

It has been assumed that the upper surface has no effect, but this can be readily calculated and allowed for. The difficulty, can, however, be avoided by placing the flat plate on the lens to be tested, supporting it on the edge by a thin layer of soft wax and pressing it down to give a central contact. If care be taken to keep the upper surface approximately level no difficulties arise.

Comparison of Two Curved Surfaces.

The same method can be employed to find the difference of curvature for the case of a convex and a concave curve giving central contact.

The procedure is as before, except that the difference of curvature,

$$R_1-R_2=\frac{(n_2-n_1)\lambda}{C_2^2-C_1^2},$$

where n_2 and n_1 are the numbers of the rings chosen. The number of rings chosen depends on the conditions of visibility, but, as before, it is often convenient to make a measurement for a difference of 17 rings to facilitate the calculation.

The apparatus used consists of a 1″ o.g. on a fairly good measuring microscope, and with sodium light the method is applicable to a difference of curvature of about $\frac{1}{400}$ mm., and gives an accuracy of about 1 in 1,000 in this case.*

As the actual curvature increases the relative accuracy in measuring the difference becomes less important, and greater differences can be measured.

When the difference of curvature is small, so that the rings become very wide, it is preferable to remove the lenses and substitute a pin hole and a cross line.

The method involves the use of standard surfaces, but in the case of steep concave curves the comparison can be made with steel balls, which are commercially procurable, and are fairly accurately spherical. Such concave surfaces can be used to check other convex surfaces. The method described above applies only to the case of central contacts. In the case of edge contacts the following method may be used :—

The standard plates are smaller than the test surface, and are accurate up to the edge (in practice they are reduced after being worked), and the method used is to determine the central separation of the two surfaces.

The difference of the optical path $n\lambda$ for the two interfering beams at the centre will depend on the angle of incidence, and as the angle is increased the path difference will change, and rings will appear to run into the centre. We count the number of rings (p) which disappear into the centre while the angle of incidence changes from i_1 to i_2.

Then

$$p=(n\cos i_1-n\cos i_2)\text{—}i.e.,\ n\lambda=\frac{p}{\cos i_1-\cos i_2}\lambda$$

and the curvature difference$=\dfrac{p\lambda}{\cos i_1-\cos i_2}\Big/C^2$,

where C is the semi-diameter of the standard plate.

The apparatus used for the preliminary measurements consists of an open sight directed towards the axis of a circle and arranged so that its angular movement can be read on a circle (or a fixed angular movement given), and a table which is adjustable to allow the rings to come to the centre of a circle.

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* In this case it is necessary to use the exact value of λ.

The standard plate is placed on the surface to be tested, and the monochromatic rings produced by a sodium flame or mercury lamp. The sight is adjusted approximately to normal incidence, and a small glass plate used to illuminate the ring system. The central ring is brought on to the sighting line; the sight is then set at a fixed angle, say 30 deg., and the rings observed; if they are not on the sighting line the table is raised or lowered. The angle of observation is then slowly increased till the angle is, say, 60 deg., the number of rings which disappear being counted. It is easy to estimate to a quarter of one ring, so that the central path difference can be found to within one wave-length.

When the number of rings becomes excessive, intermediate values of i may be used to shorten the counting.

In order to keep the illumination uniform, a finely-ground glass plate is inserted between the flame and the surfaces.

The aperture of the open sight is preferably about 1 mm. in diameter, though a slot of less width is frequently more useful, especially when the path difference is large.

Limits and Precautions.

With sodium light the double line furnishes a limit at about 500 rings on the surface—that is, the separation must not exceed about 1 mm. If necessary, smaller test surfaces may be used, but with more monochromatic sources of light this limitation is not so necessary.

When the path difference is large a small aperture must be used in the sighting device, as otherwise the angles of incidence for the various rays reaching the pupil of the eye are different, and the interference pattern becomes indistinct.

If the separation of the surfaces at the centre be h, the path difference at the centre exceeds that at the edge by $2h \cos i$, where i is the angle of incidence of the light used for the observation.

In making an observation the light used will have slightly varying angles of incidence, and the path difference will vary with the angle of incidence, so that the change di in the angle of incidence will give a change $2h \sin i di$ in the path difference, or $n\lambda \sin i di$ when $n\lambda$ is the path difference for normal incidence.

The pattern will become indistinct when $n\lambda \sin i\,di$ exceeds $\lambda/2$, so that di should not exceed $\frac{1}{2 \sin i} . \frac{1}{n}$. If $n=400$ and $i=45$ deg. di should not exceed $\frac{1}{560}$—*i.e.*, a slit $\frac{1}{2}$ mm. wide at 280 mm. distance from the pattern would suffice.

For determination of absolute curvatures it is desirable to keep the value of i_2 sufficiently large, so that the actual number of rings can be found. It is easy to estimate the number of rings which disappear to $\frac{1}{4}$ of a ring, so that where possible $\frac{1}{4}$ of the total rings should be counted, and $\cos i_1 - \cos i_2$ should equal $\frac{1}{4}$.

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